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Core stability and core-like solutions for three-sided assignment games

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Core stability and core-like solutions for three-sided assignment games

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Abstract

In this paper, we study different notions of stability for three-sided assignment games. Since the core may be empty in this case, we first focus on other notions of stability such as the notions of subsolution and von Neumann-Morgenstern stable sets. The dominant diagonal property is necessary for the core to be a stable set, and also sufficient in the case where each sector of the market has two agents. Furthermore, for any three-sided assignment market, we prove that the union of the extended cores of all μ-compatible subgames, for a given optimal matching μ, is the core with respect to those allocations that are compatible with that matching, and this union is always non-empty.

JEL: C71, C78

Keywords:

Assignment game; core; subsolution; von Neumann-Morgenstern stable set.

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Magstabilitás és mag-jellegű megoldások háromoldalú hozzárendelési játékokra

Ata Atay, Marina Núñez

Összefoglaló

Ebben a cikkben háromoldalú hozzárendelési játékokra vizsgálunk különböző stabilitási koncepciókat. Mivel a játék magja üres lehet, ezért más stabilitási fogalmakra fókuszálunk, a részmegoldásra és a von Neumann–Morgenstern-féle stabil halmazra. A domináns diagonális tulajdonság szükséges feltétele annak, hogy a mag egy stabil halmazt adjon, és elégséges is abban az esetben, amelyben minden szektorban csak két játékos van. Továbbá belátjuk azt is, hogy bármely háromoldalú hozzárendelési játékra egy adott μ optimális párosításra az összes μ-kompatibilis részjáték kibővített magjainak uniója nem üres, és megegyezik a játék magjával azon kimeneteket tekintve, amelyek kompatibilisek a μ párosítással.

JEL: C71, C78

Tárgyszavak: hozzárendelési játék, mag, részmegoldások, von Neumann–Morgenstern-féle stabil halmaz

Core stability and core-like solutions for three-sided assignment games*

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Abstract

In this paper, we study different notions of stability for three-sided assignment games. Since the core may be empty in this case, we first focus on other notions of stability such as the notions of subsolution and von Neumann-Morgenstern stable sets. The dominant diagonal property is necessary for the core to be a stable set, and also sufficient in the case where each sector of the market has two agents. Furthermore, for any three-sided assignment market, we prove that the union of the extended cores of all μ-compatible subgames, for a given optimal matching μ, is the core with respect to those allocations that are compatible with that matching, and this union is always non-empty.

Keywords: assignment game · core · subsolution · von Neumann-Morgenstern stable set

JEL classification: C71 · C78

1 Introduction

In this paper we consider markets with three different sectors or sides. Coalitions of agents can achieve a non-negative joint profit only by means of triplets if formed by one agent of each side in the market. Then, a three-dimensional valuation matrix represents the joint profit of all these possible triplets. These markets, introduced by Kaneko and

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Wooders (1982), are a generalization of Shapley and Shubik (1972) two-sided assignment games. Also, Stuart (1997) makes use of three-sided assignment markets to study a supplier-firm-buyer situation.

In a two-sided assignment game, the two sectors can be identified with a sector of buyers and a sector formed by sellers. Each seller has one unit of an indivisible good to sell and each buyer wants to buy at most one unit. Buyers have valuations over goods. The valuation matrix represents the joint profit obtained by each buyer-seller trade. From these valuations a coalitional game is obtained and the total profit under an optimal matching between buyers and sellers yields the worth of the grand coalition. The best known solution concept for coalitional game is the *core* (Gillies, 1959). Roughly speaking, a dominance relation is defined between imputations, that is, individually rational payoff vectors that distribute the worth of the grand coalition between all agents in the market. Then, the core is the set of undominated imputations.

An important difference between the two-sided and the three-sided assignment games is that while the core is always non-empty in the first case, it may be empty in the latter. This is why we are interested in the study of some other solution concept for these games.

Other well-known solution concepts for coalitional games, and hence also for assignment games, are based on the same dominance relation between imputations. A *von Neumann-Morgenstern stable set* (von Neumann and Morgenstern, 1944) is a set of imputations that satisfy internal stability and external stability: *(a)* no imputation in the set is dominated by any other imputation in the set and *(b)* each imputation outside the set is dominated by some imputation in the set. Even if its computation can be difficult, the conjecture was that all games had a stable set. However, Lucas (1968) provided an example of a game with no stable set. The core always satisfies internal stability. Moreover, the core is included in any stable set and if the core is externally stable, then it is the only stable set. An intermediate form of stability, weaker than stable sets but stronger than the core, is the notion of *subsolution* introduced by Roth (1976). Roughly speaking, a set of imputations is a subsolution if it is internally stable and it is not dominated by the set of allocations it fails to dominate. Other notions of stability are analyzed in Peris and Subiza (2013) and Han and van Deemen (2016).

In the case of two-sided assignment games, Solymosi and Raghavan (2001) shows that the core of a two-sided assignment game is a stable set if and only if the valuation matrix has a dominant diagonal. Later, Núñez and Rafels (2013) proves the existence of a stable set for all two-sided assignment games. The stable set they introduce is the only one that excludes third party payments with respect to an optimal matching μ and is defined through certain subgames, which are called μ-compatible subgames.

However, when the market has more than two sides, most results for the two-sided case do not extend to the three-sided case. Kaneko and Wooders (1982) shows that the core of a three-sided assignment game may be empty. Moreover, when the core is non-empty it fails to have a lattice structure. Lucas (1995) provides necessary and sufficient conditions that yield non-emptiness of the core for the particular case where each side of the market consists of two agents. Nonetheless, there are no results on stable sets for three-sided assignment games.

The fact that the core may be empty makes the notions of subsolution and of stable sets more appealing as a solution concept for three-sided assignment games.

First, we generalize the notion of dominant diagonal to the three-sided case and prove that it is a necessary condition for the core of this game to be a stable set. We also show that for three-sided markets with only two agents on each side, the dominant diagonal property suffices to guarantee that the core is stable. Furthermore, we extend the notion of μ-compatible subgames introduced by Núñez and Rafels (2013) to the three-sided case. As a consequence, given a three-sided game and an optimal matching μ, we consider the set V^{μ} formed by the union of the cores of all μ-compatible subgames. However, different to the two-sided case, we show by means of a counterexample that V^{μ} may not be a stable set, not even a subsolution.

A second approach to our problem will be to modify the definition of the "core". Although the usual definition of the core and the stable sets of a coalitional game takes as the set of feasible outcomes of the game the set of imputations (efficient allocations that are individually rational), a more general setting can be considered. Lucas (1992) defines an abstract game by a set of (feasible) outcomes B and a dominance relation D (irreflexive binary relation) over this set of outcomes. Then, the core of an abstract game is the set of undominated outcomes, $C = B \setminus D(B)$, and a stable set V is a set of outcomes such that $V = B \setminus D(V)$.

In a three-sided assignment game it seems natural to restrict the set of feasible outcomes to those imputations that are compatible with some optimal matching μ, that is, allocations where the only side-payments take place within the triplets μ. These allocations are known as the principal section B^{μ} of the assignment game and we prove that the set V^{μ} introduced before is the set of undominated allocations: $V^{\mu} = B^{\mu} \setminus D(B^{\mu})$. In this sense, V^{μ}, which is always non-empty, is the "core" with respect to the principal section. Moreover, V^{μ} coincides with the usual core if and only if the valuation matrix has a dominant diagonal.

The paper is organized as follows. In Section 2 we give preliminaries on assignment games and solution concepts. Section 3 is devoted to conditions on the three-sided valuation matrix in order to obtain core stability. In Section 4, μ-compatible subgames are introduced and the union of cores of all μ-compatible subgames is shown to coincide with the core if the valuation matrix has a dominant diagonal. Finally, in Section 5, we show that if the μ-principal section is considered as the set of feasible outcomes, the union of the cores of all μ-compatible subgames, V^{μ}, is the set of undominated outcomes, that is, the "core" with respect to the set of feasible outcomes.

2 Preliminaries

Let $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ be pairwise disjoint countable sets. An $m \times m \times m$ *assignment market* $\gamma = (M_1, M_2, M_3; A)$ consists of three different sectors with m agents each: $M_1 = \{1, 2, ..., m\} \subseteq \mathcal{U}_1$, $M_2 = \{1', 2', ..., m'\} \subseteq \mathcal{U}_2$, $M_3 = \{1'', 2'', \ldots, m''\} \subseteq \mathcal{U}_3$, and a three-dimensional valuation matrix $A = (a_{ijk})_{\substack{i\in M_1\\ j\in M_2\\ k\in M_3}}$ that represents the potential joint profit obtained by triplets formed by one agent of each sector. These triplets are the *basic coalitions* of the three-sided assignment game, as defined by Quint (1991).

Given subsets of agents of each sector, $S_1 \subseteq M_1$, $S_2 \subseteq M_2$, and $S_3 \subseteq M_3$, a *matching* μ for the submarket $\gamma_{|S} = (S_1, S_2, S_3; A_{|S_1\times S_2\times S_3})$ is a subset of the cartesian product,

$\mu \subseteq S_1 \times S_2 \times S_3$, such that each agent belongs to at most one triplet. We denote by $\mathcal{M}(S_1, S_2, S_3)$ the set of all possible matchings. A matching $\mu \in \mathcal{M}(S_1, S_2, S_3)$ is an *optimal matching* for the submarket if

$$\sum_{(i,j,k)\in\mu} a_{ijk} \geq \sum_{(i,j,k)\in\mu'} a_{ijk}$$

for all other $\mu' \in \mathcal{M}(S_1, S_2, S_3)$. We denote by $\mathcal{M}_A(S_1, S_2, S_3)$ the set of all optimal matchings for the submarket $(S_1, S_2, S_3; A_{|S_1\times S_2\times S_3})$.

The $m \times m \times m$ *assignment game*, (N, w_A), related to the above assignment market has player set $N = M_1 \cup M_2 \cup M_3$ and characteristic function

$$w_A(S) = \max_{\mu\in\mathcal{M}(S\cap M_1, S\cap M_2, S\cap M_3)} \sum_{(i,j,k)\in\mu} a_{ijk}$$

for all $S \subseteq N$. In the sequel, we will need to exclude some agents. Then, if we exclude some agents $I \subseteq M_1$, $J \subseteq M_2$, and $K \subseteq M_3$, we will write $w_{A_{-I\cup J\cup K}}$ instead of $w_{A_{|(M_1\setminus I)\times(M_2\setminus J)\times(M_3\setminus K)}}$. Notice that these subgames need not have the same number of agents in each sector. Nevertheless, the notion of matching and characteristic function is defined analogously as for the $m \times m \times m$ case.

Given an $m\times m\times m$ assignment game, a payoff vector, or an allocation, is $(u, v, w) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+ \times \mathbb{R}^m_+$ where u_l denotes the payoff[1] to agent $l \in M_1$, v_l denotes the payoff to agent $l' \in M_2$ and w_l denotes the payoff to agent $l'' \in M_3$. An *imputation* is a non-negative payoff vector that is efficient, $u(M_1) + v(M_2) + w(M_3) = \sum_{i\in M_1} u_i + \sum_{j\in M_2} v_j + \sum_{k\in M_3} w_k = w_A(M_1 \cup M_2 \cup M_3)$. We denote the set of imputations of the assignment game (N, w_A) by $I(w_A)$.

Given an optimal matching $\mu \in \mathcal{M}_A(M_1, M_2, M_3)$ we define the *μ-principal section* of (N, w_A), as the set of payoff vectors such that $u_i + v_j + w_k = a_{ijk}$ for all $(i, j, k) \in \mu$ and the payoff to agents unassigned by μ is zero. We denote it by $B^\mu(w_A)$. Notice that $B^\mu(w_A) \subseteq I(w_A)$. In the μ-principal section the only side payments that take place are those among agents matched together by μ.

We can assume that the optimal matching is on the main diagonal of the valuation matrix, $\mu = \{(i, i', i'')|i \in \{1, 2, \ldots, m\}\}$. Notice that the allocation $(a, 0, 0)$, that is $u_i = a_{iii}$ for all $i \in M_1$, $v_j = w_k = 0$ for all $j \in M_2, k \in M_3$, always belongs to the μ-principal section. The same happens with the allocations $(0, a, 0)$ and $(0, 0, a)$. These three vertices of the polytope $B^\mu(w_A)$ will be named the *sector-optimal allocations*. The core of a game is the set of imputations (u, v, w) such that no coalition S can improve upon: $u(S \cap M_1) + v(S \cap M_2) + w(S \cap M_3) \geq w_A(S)$. In our case, it is easy to see that it is enough to consider individual and basic coalitions. An imputation (u, v, w) belongs to the core, $(u, v, w) \in C(w_A)$, if and only if for all $(i, j, k) \in M_1 \times M_2 \times M_3$ it holds $u_i + v_j + w_k \geq a_{ijk}$ and $u_i \geq 0$ for all $i \in M_1$, $v_j \geq 0$ for all $j \in M_2$, and $w_k \geq 0$ for all $k \in M_3$. Notice that, together with efficiency, the above constraints imply that the core is a subset of the μ-principal section for any optimal matching $\mu \in \mathcal{M}_A(M_1, M_2, M_3)$.

It is well-known (see Kaneko and Wooders, 1982) that the core of a three-sided assignment game may be empty. For the particular case where each sector contains only

[1] $\mathbb{R}^m_+$ is the set of non-negative real numbers. Hence, to simplify notation, we only consider individually rational payoff vectors.

two agents, Lucas (1995) gives necessary and sufficient conditions for balancedness (that is non-emptiness of the core). Under the assumption that $\mu = \{(1, 1', 1''), (2, 2', 2'')\}$ is an optimal matching, the core of a $2 \times 2 \times 2$ assignment game is non-empty if and only if it satisfies the following conditions:

$$\begin{aligned} 2a_{111} + a_{222} &\geq a_{112} + a_{121} + a_{211}, \\ a_{111} + 2a_{222} &\geq a_{221} + a_{212} + a_{122}. \end{aligned} \tag{1}$$

Given a three-sided assignment market $\gamma = (M_1, M_2, M_3; A)$, we define a binary relation on the set of imputations. It is called the *dominance relation*. Given two imputations (u, v, w) and (u', v', w'), we say (u, v, w) *dominates* (u', v', w') if and only if there exists $(i, j, k) \in M_1 \times M_2 \times M_3$ such that $u_i > u'_i$, $v_j > v'_j$, $w_k > w'_k$ and $u_i + v_j + w_k \leq a_{ijk}$. We denote it by (u, v, w) dom ${}^{A}_{\{i,j,k\}}(u', v', w')$. We write (u, v, w) dom ${}^{A}(u', v', w')$ to denote that (u, v, w) dominates (u', v', w') by means of some triplet (i, j, k).[2] Given a set of imputations $V \subseteq I(w_A)$, we denote by $D(V)$ the set of imputations dominated by some element in V and by $U(V)$ those imputations not dominated by any element in V.

Two main set-solution concepts are defined by means of this dominance relation: the core and the stable set. On the one side, the core, whenever it is non-empty, coincides with the set of undominated imputations. That is, $C(w_A) = U(I(w_A))$. The other solution concept defined by means of domination is the von Neumann-Morgenstern stable set.

A subset of the set of imputations, $V \subseteq I(w_A)$, is a *von Neumann-Morgenstern solution* or a *stable set* if it satisfies internal and external stability:

(i) *internal stability:* for all $(u, v, w), (u', v', w') \in V$, $(u, v, w)\ \text{dom}^A(u', v', w')$ does not hold,

(ii) *external stability:* for all $(u', v', w') \in I(w_A) \setminus V$, there exists $(u, v, w) \in V$ such that $(u, v, w)\ \text{dom}^A(u', v', w')$.

Internal stability of a set of imputations V guarantees that no imputation of V is dominated by another imputation of $V : V \subseteq U(V)$. The core is internally stable. External stability imposes that all imputations outside V are dominated by an imputation in $V : I(w_A) \setminus V \subseteq D(V)$. In general, the core fails to satisfy external stability. Both conditions (internal and external stability) can be summarized in $V = U(V)$.

There is an intermediate notion of stability introduced by Roth (1976). A subset of imputations $V \subseteq I(w_A)$ is a *subsolution* if

(i) V is internally stable, that is, $V \subseteq U(V)$,

(ii) $V = U^2(V) = U(U(V))$.

[2] This dominance relation is the usual one introduced by von Neumann and Morgenstern (1944). It is clear that in the case of multi-sided assignment games, we only need to consider domination via basic coalitions. When no confusion regarding the valuation matrix can arise, we will simply write (u, v, w) dom (u', v', w').

Together with the internal stability, that the concept of subsolution shares with the core and the stable sets, the second condition for a set V to be a subsolution requires that if an imputation $x \in V$ is dominated by some $y \notin V$, then y will be dominated by some other $z \in V$. Notice that this is like an external stability restricted to those external imputations that dominate some element of V. In this sense, this stability notion is weaker than that of stable sets. For arbitrary coalitional games, Roth (1976) proves a subsolution always exists but the existence of a non-empty subsolution is not guaranteed.

Since for three-sided assignment games the core may be empty, our first attempt is to investigate whether the stable set obtained in Núñez and Rafels (2013) for two-sided assignment games can be extended to the three-sided case. To this end, we first analyse under which conditions the core of a three-sided assignment game is already a stable set.

3 Dominant diagonal and core stability

In this section we look for conditions on the three-sided valuation matrix that guarantee the core satisfies external stability and hence it is a von Neumann-Morgenstern stable set.

We begin by generalizing to the three-sided case the dominant diagonal property introduced by Solymosi and Raghavan (2001) for two-sided assignment games. They prove that, in the two-sided case, this condition characterizes stability of the core. Therefore, we must define the appropriate generalization. We will assume that the valuation matrix is square, that is, there is the same number of agents on each side. Notice that, whenever necessary, we can assume without loss of generality that an optimal matching is placed on the main diagonal.

Definition 1. Let $(M_1 \cup M_2 \cup M_3, w_A)$ be a three-sided assignment game with $|M_1| = |M_2| = |M_3| = m$. Matrix A has a *dominant diagonal* if and only if for all $i \in \{1, 2, ..., m\}$ it holds

$$a_{iii} \geq \max\{a_{ijk}, a_{jik}, a_{jki}\} \quad \text{for all } j, k \in \{1, 2, ..., m\}.$$

Clearly, if A has a dominant diagonal, then $\mu = \{(i, i', i'') \mid i \in \{1, 2, \ldots, m\}\}$ is an optimal matching.

As in the two-sided case, the dominant diagonal property characterizes those markets where giving the profit of each optimal partnership to one given sector leads to a core element.

Proposition 2. *Let $(M_1 \cup M_2 \cup M_3, w_A)$ be a three-sided assignment game with $|M_1| = |M_2| = |M_3| = m$. The valuation matrix A has a* dominant diagonal *if and only if all sector-optimal allocations belong to the core.*

Proof. First, we prove the "*if*" part. Take the sector-optimal allocation for the first sector: $(u, v, w) = (a_{111}, ..., a_{mmm}; 0, ..., 0; 0, ..., 0)$. If it belongs to the core, then we have $a_{iii} = u_i = u_i + v_j + w_k \geq a_{ijk}$ for all $(i, j, k) \in M_1 \times M_2 \times M_3$. For the rest of optimal allocations the proof is analogous.

To prove the "*only if*" part, let A be a three-dimensional valuation matrix with the dominant diagonal property. By Definition 1, for all $i \in \{1, 2, ..., m\}$ and for all $j, k \in \{1, 2, ..., m\}$, $a_{iii} \geq \max\{a_{ijk}, a_{jik}, a_{jki}\}$. If we take the sector-optimal allocation $(u, v, w) = (a_{111}, ..., a_{mmm}; 0, ..., 0; 0, ..., 0)$, the above inequality trivially shows that it belongs to the core. Analogously, $(0, a, 0)$ and $(0, 0, a)$ are also core allocations. □

The above proposition provides a characterization of the dominant diagonal property. Since the fact that the sector-optimal core allocations belong to the core does not depend on the selected optimal matching, the dominant diagonal property is also independent of the optimal matching that is placed on the main diagonal.

Next proposition shows that the dominant diagonal property is necessary for the stability of the core.

Proposition 3. *Let $(M_1 \cup M_2 \cup M_3, w_A)$ be a three-sided assignment game with $|M_1| = |M_2| = |M_3| = m$, and an optimal matching on the main diagonal. The core is a von Neumann-Morgenstern stable set, then its corresponding valuation matrix A has a dominant diagonal.*

Proof. Let us suppose, on the contrary, that the core of a three-sided assignment game (N, w_A) is a von Neumann-Morgenstern stable set but its corresponding three-dimensional valuation matrix A has not a dominant diagonal. If A has not a dominant diagonal, then there exists one sector-optimal allocation, let us say $(a, 0, 0)$, that does not belong to the core. But then, since $C(w_A)$ is assumed to be a von Neumann-Morgenstern stable set, there exists $(u', v', w') \in C(w_A)$ such that

$$(u', v', w') \text{ dom}_{\{i,j,k\}}(a, 0, 0).$$

Then, $u'_i > u_i = a_{iii}$ which contradicts $(u', v', w') \in C(w_A)$. □

Proposition 3 rises the question of the equivalence between the von Neumann-Morgenstern stability of the core and the dominant diagonal property of the matrix. That is to say, if A has dominant diagonal, is the core of the assignment game, $C(w_A)$, a von Neumann-Morgenstern stable set? We can answer this question affirmatively when the market has only two agents in each sector. The proof is consigned to the Appendix A.

Proposition 4. *Given a $2 \times 2 \times 2$ assignment game (N, w_A) with an optimal matching on the main diagonal, the core $C(w_A)$ is a von Neumann-Morgenstern stable set if and only if A has a dominant diagonal.*

Now, we return to the general case, that is to say, $m \times m \times m$ assignment games, and define μ-compatible subgames in search of a stable set. We give some results related to stability but we do not achieve a characterization or an existence theorem.

4 The μ-compatible subgames

In this section, we follow an approach similar to the one in Núñez and Rafels (2013) to construct a stable set for two-sided assignment markets. First, we extend to three-sided assignment games the notion of the μ-compatible subgame. Then, we introduce a set that consists of the union of the extended cores of all μ-compatible subgames and we look for stability properties of this set. We show that, in general, it fails to satisfy external stability and hence, different from the two-sided case, it does not always result in a von Neumann-Morgenstern stable set.

Definition 5. Let $(M_1 \cup M_2 \cup M_3, w_A)$ be a three-sided assignment game, with $m = |M_1| = |M_2| = |M_3|$, $\mu \in \mathcal{M}_A(M_1, M_2, M_3)$ an optimal matching, and $I \subseteq M_1, J \subseteq M_2$ and $K \subseteq M_3$. The subgame

$$(M_1 \setminus I, M_2 \setminus J, M_3 \setminus K, w_{A_{-I \cup J \cup K}})$$

is a *μ-compatible subgame* if and only if

$$\begin{aligned} w_A(M_1 \cup M_2 \cup M_3) &= w_A((M_1 \setminus I) \cup (M_2 \setminus J) \cup (M_3 \setminus K)) \\ &+ \sum_{\substack{(i,j,k)\in\mu \\ i \in I}} a_{ijk} + \sum_{\substack{(i,j,k)\in\mu \\ j \in J}} a_{ijk} + \sum_{\substack{(i,j,k)\in\mu \\ k \in K}} a_{ijk}. \end{aligned}$$

When a subgame is μ-compatible, each agent outside the subgame can leave the market with the full profit of his/her partnership in the optimal matching μ, and what remains is exactly the worth of the resulting submarket. As a consequence, any core element of the subgame can be completed with the payoffs of the excluded agents to obtain an imputation of the initial market.

Without loss of generality, assume that the diagonal matching is an optimal matching for A: $\mu = \{(i, i', i'') | i \in \{1, 2, \dots, m\}\}$. Then, given a μ-compatible subgame $w_{A_{-I \cup J \cup K}}$ we define its extended core,

$$\hat{C}(w_{A_{-I \cup J \cup K}}) = \left\{ (x, z) \in B^{\mu}(w_A) \,\middle|\, \begin{array}{l} x_i = a_{iii} \text{ for all } i \in I \cup J \cup K, \\ z \in C(w_{A_{-I \cup J \cup K}}) \end{array} \right\}.$$

Note that if $C(w_{A_{-I \cup J \cup K}}) = \emptyset$, then $\hat{C}(w_{A_{-I \cup J \cup K}}) = \emptyset$. The following ones are two straightforward properties of μ-compatible subgames.

If $w_{A_{-I \cup J \cup K}}$ is a μ-compatible subgame, then:

(i) The restriction of μ is optimal for the subgame: $\mu_{|(M_1 \setminus I) \times (M_2 \setminus J) \times (M_3 \setminus K)} = \{(i, j, k) \in \mu \mid i \in M_1 \setminus I, j \in M_2 \setminus J, k \in M_3 \setminus K\}$ is an optimal matching for $w_{A_{-I \cup J \cup K}}$, which implies that the partners of agents in $I \cup J \cup K$ remain unmatched in the subgame,

(ii) if $i, j \in I \cup J \cup K$, then i and j cannot belong to the same basic coalition in μ except if the value of this triplet is null.

Hence, if $A > 0$, that is all entries are positive, all μ-compatible subgames come from the exclusion of a set of agents of only one side of the market. In particular, if we exclude all agents in M_1, then the game $(N \setminus M_1, w_{A_{-M_1}})$ is always a μ-compatible subgame since $w_{A_{-M_1}}(N \setminus M_1) = 0$. The core of this μ-compatible subgame is reduced to $\{(0, 0)\} \subseteq \mathbb{R}^{M_2} \times \mathbb{R}^{M_3}$ and the corresponding extended core is $\hat{C}(w_{A_{-M_1}}) = \{(a, 0, 0)\}$. Analogous μ-compatible subgames are obtained when we exclude the agents of one of the remaining sides of the market.

Given a three-sided assignment market $\gamma = (M_1, M_2, M_3; A)$, we define the set of all coalitions that give rise to μ-compatible subgames:

$$\mathcal{C}^\mu(A) = \{R \subseteq M_1 \cup M_2 \cup M_3 \mid w_{A_{-R}} \text{ is a } \mu\text{-compatible subgame}\}.$$

Notice that when $R = \emptyset$ we retrieve the core of the initial game (N, w_A).

Now, for any assignment market $\gamma = (M_1, M_2, M_3; A)$, we define the set $V^\mu(w_A)$ formed by the union of extended cores of all μ-compatible subgames:

$$V^\mu(w_A) = \bigcup_{R \in \mathcal{C}^\mu(A)} \hat{C}(w_{A_{-R}}) \tag{2}$$

A first immediate consequence of the above definition is that $V^\mu(w_A)$ is a subset of the μ-principal section:

$$V^\mu(w_A) \subseteq B^\mu(w_A).$$

Notice also that differently from the core, the set $V^\mu(w_A)$ is always non-empty since it contains at least the three points $(a, 0, 0)$, $(0, a, 0)$, and $(0, 0, a)$, which result from the μ-compatible subgames where all agents of one sector have been excluded. In fact the following example shows that $V^\mu(w_A)$ can be reduced to only these three points and hence be non-convex and disconnected.

Example 6. Consider a three-sided assignment game where each sector has two agents, $M_1 = \{1, 2\}$, $M_2 = \{1', 2'\}$, and $M_3 = \{1'', 2''\}$, and the valuation matrix A is the following

$$A = \begin{matrix} & \begin{matrix} 1' & 2' \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{pmatrix} \mathbf{3} & 1 \\ 2 & 5 \end{pmatrix} \\ & 1'' \end{matrix} \qquad \begin{matrix} & \begin{matrix} 1' & 2' \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{pmatrix} 1 & 4 \\ 5 & \mathbf{4} \end{pmatrix} \\ & 2'' \end{matrix}$$

Notice there is a unique optimal matching $\mu = \{(1, 1', 1''), (2, 2', 2'')\}$. By Lucas' conditions for balancedness, see (1), we notice that the core is empty: $a_{111} + 2a_{222} = 11 < 14 = a_{221} + a_{122} + a_{212}$. We observe that the only μ-compatible subgames are $w_{A_{-\{1,2\}}}$, $w_{A_{-\{1',2'\}}}$ and $w_{A_{-\{1'',2''\}}}$. Hence $V^\mu(w_A) = \{(a, 0, 0), (0, a, 0), (0, 0, a)\} = \{(3, 4; 0, 0; 0, 0), (0, 0; 3, 4; 0, 0), (0, 0; 0, 0; 3, 4)\}$. Now it is easy to realize that such points do not dominate any imputation in the μ-principal section. Thus, external stability does not hold for the set $V^\mu(w_A)$. This implies that the set $V^\mu(w_A)$ is not a von Neumann-Morgenstern stable set.

Now, take the imputation $(1, 4.5; 1, 0.25; 0.25, 0)$. Notice that it is not an element of the set $V^\mu(w_A)$ and there is no element of the set $V^\mu(w_A)$ that dominates it. Furthermore, it dominates an element, $(3, 4; 0, 0; 0, 0)$, of the set $V^\mu(w_A)$ via coalition $\{2, 2', 1''\}$. Hence, there exist an imputation that dominates one allocation in $V^\mu(w_A)$ and no point in $V^\mu(w_A)$ dominates the aforementioned allocation, which shows the set $V^\mu(w_A)$ is not a subsolution.

The following proposition provides an equivalent definition of the set $V^\mu(w_A)$.

Proposition 7. *Let $(M_1 \cup M_2 \cup M_3, w_A)$ be a three-sided assignment game with $|M_1| = |M_2| = |M_3| = m$, and an optimal matching μ on the main diagonal. Let (u, v, w) be an allocation of the principal section, that is, $(u, v, w) \in B^\mu(w_A)$. Then $(u, v, w) \in V^\mu(w_A)$ if and only if for all $(i, j, k) \in M_1 \times M_2 \times M_3$ at least one of the four following statements holds:*

(i) either $u_i = a_{iii}$

(ii) or $v_j = a_{jjj}$

(iii) or $w_k = a_{kkk}$

(iv) or $u_i + v_j + w_k \geq a_{ijk}$.

Proof. First, we prove the "*only if*" part. Assume $(u, v, w) \in \hat{C}(w_{A_{-R}})$ for some $R \subseteq M_1 \cup M_2 \cup M_3$ and take $(i, j, k) \in M_1 \times M_2 \times M_3$. If $i \in R$, then $u_i = a_{iii}$. If $j \in R$, then $v_j = a_{jjj}$. If $k \in R$, then $w_k = a_{kkk}$. Otherwise, $u_i + v_j + w_k \geq a_{ijk}$.

Next, we show the "*if*" implication. Take $(u, v, w) \in B^\mu(w_A)$ such that all $(i, j, k) \in M_1 \times M_2 \times M_3$ satisfy either (i), or (ii), or (iii), or (iv). Define $I = \{i \in M_1 \mid u_i = a_{iii}\}$, $J = \{j \in M_2 \mid v_j = a_{jjj}\}$, and $K = \{k \in M_1 \mid w_k = a_{kkk}\}$, and also $R = I \cup J \cup K$. Notice that $z = (u, v, w) \in \hat{C}(w_{A_{-R}})$, since $z_l = a_{lll}$ for all $l \in R$, and for all $(i, j, k) \in (M_1 \setminus R) \times (M_2 \setminus R) \times (M_3 \setminus R)$ it holds $u_i + v_j + w_k \geq a_{ijk}$. Hence, $z = (u, v, w) \in V^\mu(w_A)$. □

The above proposition shows that the allocations in the set $V^\mu(w_A)$ satisfy all core constraints except maybe those constraints involving an agent that is paid the full profit of his/her partnership in μ.

Making use of the above equivalent expression of the set $V^\mu(w_A)$, we can characterize under which condition this set reduces to the core of the three-sided assignment market.

Proposition 8. *Let $(M_1 \cup M_2 \cup M_3, w_A)$ be a three-sided assignment game with $|M_1| = |M_2| = |M_3|$ and μ an optimal matching on the main diagonal, $\mu \in \mathcal{M}_A(M_1, M_2, M_3)$. A has a dominant diagonal if and only if $V^\mu(w_A) = C(w_A)$.*

Proof. First, we prove the "*if*" part. Taking $R = M_1$ always gives a μ-compatible subgame and $\hat{C}(w_{A_{-M_1}}) = \{(a, 0, 0)\}$. Then, by the assumption, $(a, 0, 0) \in C(w_A)$. Similarly, $(0, a, 0) \in C(w_A)$ and $(0, 0, a) \in C(w_A)$. By Proposition 2, we obtain that A has dominant diagonal.

To prove the "*only if*" part, assume $(u, v, w) \in \hat{C}(w_{A_{-R}})$. Since $\hat{C}(w_{A_{-R}}) \subsetneq B^\mu(w_A)$, (u, v, w) satisfies the efficiency condition. By the definition of the extended core, we know that, for all $i \in R \cap M_1$, $u_i = a_{iii}$; for all $j \in R \cap M_2$, $v_j = a_{jjj}$; for all $k \in R \cap M_3$, $w_k = a_{kkk}$; and for all $(i, j, k) \in (M_1 \setminus R) \times (M_2 \setminus R) \times (M_3 \setminus R)$ it satisfies $u_i + v_j + w_k \geq a_{ijk}$. Now, if $i \in R$, for all $j \in M_2$ and $k \in M_3$ it holds $u_i + v_j + w_k = a_{iii} + v_j + w_k \geq a_{iii} \geq a_{ijk}$, where the last inequality follows from the dominant diagonal property. Similarly, if $j \in R$ and $i \in M_1$, $k \in M_3$ or $k \in R$ and $i \in M_1$, $j \in M_2$ we obtain $u_i + v_j + w_k \geq a_{ijk}$. Together with efficiency this means $(u, v, w) \in C(w_A)$. □

We have seen that in general the set $V^\mu(w_A)$ is not a stable set nor a subsolution, but it is always a non-empty set. In the next section, we give a characterization of the set $V^\mu(w_A)$ by means of the dominance relation.

5 The core of a three-sided assignment game with respect to the principal section

We have just seen that under the dominant diagonal property the set $V^\mu(w_A)$ coincides with the core and hence it is the set of undominated imputations.

In an assignment market, once an optimal matching μ is agreed on, agents must negotiate on an outcome that distributes the profit of each optimally matched triplet among its members. That is to say, it seems natural to consider payoff vectors that exclude side-payments among agents that are not in the same optimal triplet. These payoff vectors are those in the μ-principal section $B^\mu(w_A)$.

Next theorem shows that, if we reduce to the outcomes in the μ-principal section, the set $V^\mu(w_A)$ is precisely the set of undominated outcomes, even if the dominant diagonal property does not hold.

Theorem 9. *Let $(M_1 \cup M_2 \cup M_3, w_A)$ be a three-sided assignment game with $|M_1| = |M_2| = |M_3| = m$, and $\mu \in \mathcal{M}_A(M_1, M_2, M_3)$. Then,*

$$V^\mu(w_A) = U(B^\mu(w_A))$$

where $U(B^\mu(w_A))$ is the set of imputations that are undominated by the μ-principal section.

Proof. Let us write $V = V^\mu(w_A)$ and assume μ is on the main diagonal. First, we prove $U(B^\mu(w_A)) \subseteq B^\mu(w_A)$. Notice that this inclusion is equivalent to $I(w_A) \setminus B^\mu(w_A) \subseteq D(B^\mu(w_A))$, where $D(B^\mu(w_A))$ is the set of imputations that are dominated by some allocation in the μ-principal section.

Take $(x, y, z) \in I(w_A) \setminus B^\mu(w_A)$. Then, there exists $i \in \{1, ..., m\}$ such that $x_i + y_i + z_i < a_{iii}$. Take $\varepsilon = a_{iii} - x_i - y_i - z_i > 0$, and define λ_1, λ_2 and λ_3 by $\lambda_1 = \frac{x_i + \frac{\varepsilon}{3}}{a_{iii}}$, $\lambda_2 = \frac{y_i + \frac{\varepsilon}{3}}{a_{iii}}$ and $\lambda_3 = \frac{z_i + \frac{\varepsilon}{3}}{a_{iii}}$. Note that $\lambda_1 + \lambda_2 + \lambda_3 = 1$, and $\lambda_1 a_{iii} = x_i + \frac{\varepsilon}{3} > x_i$, $\lambda_2 a_{iii} = y_i + \frac{\varepsilon}{3} > y_i$ and $\lambda_3 a_{iii} = z_i + \frac{\varepsilon}{3} > z_i$.

Now, recall that $(a, 0, 0)$, $(0, a, 0)$ and $(0, 0, a)$ all belong to $B^\mu(w_A)$ and take the point $(u, v, w) = \lambda_1(a, 0, 0) + \lambda_2(0, a, 0) + \lambda_3(0, 0, a) \in B^\mu(w_A)$. Then, for all $i \in \{1, ..., m\}$, $u_i + v_i + w_i = (\lambda_1 + \lambda_2 + \lambda_3)a_{iii} = a_{iii}$. Together with $u_i > x_i$, $v_i > y_i$ and $w_i > z_i$, this implies that (u, v, w) dom $_{\{i,i',i''\}}(x, y, z)$ and hence $(x, y, z) \in D(B^\mu(w_A))$.

Now, we prove the equality, $V = U(B^\mu(w_A))$. First, we prove $V \subseteq U(B^\mu(w_A))$. We want to show that no allocation in V is dominated by an allocation in the μ-principal section. Consider two allocations $(u, v, w) \in B^\mu(w_A)$ and $(u', v', w') \in V$. We want to show that (u, v, w) cannot dominate (u', v', w') via any triplet $\{i, j, k\}$. Assume that for some $(i, j, k) \in M_1 \times M_2 \times M_3$, (u, v, w) dom$_{\{i,j,k\}}$ (u', v', w') holds, which means $u_i + v_j + w_k \leq a_{ijk}$ together with $u_i > u'_i$, $v_j > v'_j$ and $w_k > w'_k$. Two cases are considered.

Case 1: $(u', v', w') \in C(w_A)$.

We reach straightforwardly a contradiction, since core elements are undominated.

Case 2: $(u', v', w') \in \hat{C}(w_{A_{-R}})$ for some $R \in \mathcal{C}^\mu(A)$.

If $i \in R$, then $u'_i = a_{iii}$. Then $u_i > u'_i = a_{iii}$ which contradicts $(u, v, w) \in B^\mu(w_A)$. The same argument leads to contradiction if $j \in R$ or $k \in R$. If $i \notin R$, $j \notin R$ and $k \notin R$, then by Proposition 7, $u'_i+v'_j+w'_k \geq a_{ijk} \geq u_i+v_j+w_k$ which contradicts our assumption $u_i > u'_i$, $v_j > v'_j$ and $w_k > w'_k$. This finishes the proof of $(u,v,w) \in U(B^\mu(w_A))$.

Now, we move to $U(B^\mu(w_A)) \subseteq V$. Assume on the contrary that $(u,v,w) \in U(B^\mu(w_A))$ and $(u,v,w) \notin V$. Since $U(B^\mu(w_A)) \subseteq B^\mu(w_A)$, $(u,v,w) \in B^\mu(w_A)$. Then, $(u,v,w) \in B^\mu(w_A)$ and $(u,v,w) \notin V$ which implies by Proposition 7 there exist $(i,j,k) \in M_1 \times M_2 \times M_3$ such that $u_i < a_{iii}$, $v_j < a_{jjj}$, $w_k < a_{kkk}$ and $u_i + v_j + w_k < a_{ijk}$. Define $\varepsilon_1 = a_{iii} - u_i > 0$, $\varepsilon_2 = a_{jjj} - v_j > 0$, $\varepsilon_3 = a_{kkk} - w_k > 0$ and $\varepsilon_4 = a_{ijk} - u_i - v_j - w_k > 0$. Also, let us define $u'_i = u_i + \min\{\varepsilon_1, \frac{\varepsilon_4}{3}\}$, $v'_j = v_j + \min\{\varepsilon_2, \frac{\varepsilon_4}{3}\}$ and $w'_k = w_k + \min\{\varepsilon_3, \frac{\varepsilon_4}{3}\}$. Note that $u'_1 > u_1$, $v'_j > v_j$, $w'_k > w_k$ and $u'_i + v'_j + w'_k < u_i + v_j + w_k + 3\frac{\varepsilon_4}{3} = a_{ijk}$. Now, we complete the definition of $(u', v,' w')$ in the following way:

Since, by definition, $u'_i \leq a_{iii}$, define $v'_i = a_{iii} - u'_i$ and $w'_i = 0$. Similarly, since $v'_j \leq a_{jjj}$, define $u'_j = a_{jjj} - v'_j$ and $w'_i = 0$. And finally, since $w'_k \leq a_{kkk}$, define $v'_k = a_{kkk} - w'_k$ and $u'_k = 0$. For all $l \in \{1, ..., m\} \setminus \{i,j,k\}$ define $u'_l = a_{lll}$, $v'_l = 0$ and $w'_l = 0$. Then $(u', v', w') \in B^\mu(w_A)$ and (u', v', w') dom $_{\{i,j,k\}}(u,v,w)$ which contradicts $(u,v,w) \in U(B^\mu(w_A))$. Hence, if $(u,v,w) \in U(B^\mu(w_A))$, then $(u,v,w) \in V$. □

In Theorem 9 we show that there is no allocation in the μ-principal section that dominates any element of $V^\mu(w_A)$. This ensures internal stability of $V^\mu(w_A)$. But, we already know from Example 6 that $V^\mu(w_A)$ may not be externally stable. Hence, it may not be a stable set. We have not been able to prove existence of stable sets for three-sided assignment games. However, if given an optimal matching μ, there existed a stable set included in the μ-principal section, then $V^\mu(w_A)$ would be included in this stable set. Recall also that $V^\mu(w_A)$ contains the core $C(w_A)$ and $V^\mu(w_A) = C(w_A)$ whenever A has a dominant diagonal.

Moreover, the sets $V^\mu(w_A)$, one for each optimal matching, have several appealing properties. They are always non-empty and moreover, if the set of feasible outcomes is not the whole imputation set but the μ-principal section $B^\mu(w_A)$, for some optimal matching μ, then $V^\mu(w_A)$ is the set of undominated allocations. Hence, when no side payments take place except those among the agents in an optimally matched triplet, then $V^\mu(w_A)$ is like the "core" with respect to this set of feasible payoff sectors.

Even if we do not restrict to the μ-principal section, that is we allow for side payments among agents not matched together and consider any imputation as a feasible outcome, the set $V^\mu(w_A)$ has still an appealing economic interpretation. In any allocation in $V^\mu(w_A)$, there may be some agents strong enough to require the whole profit of their optimal partnership. Their partners cannot prevent them from doing so since these partners will remain unmatched when these strong players leave the market. Finally, the remaining agents agree on a core allocation of the resulting subgame.

A Appendix: the $2 \times 2 \times 2$ Case

In this appendix, we show that the property of dominant diagonal is also a sufficient condition for core stability in the particular case of three-sided assignment games with two agents in each side. To this end, we need a remark regarding 2×2 assignment games that will be of use in the proof of Proposition 4.

Remark 10. Let $(M \cup M', w_B)$ be a 2×2 assignment game with $M = \{1, 2\}$, $M' = \{1', 2'\}$, and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$. Let us denote by $(\bar{u}, \underline{v})$, the buyers-optimal core allocation, that is, each buyer maximizes his/her payoff while each seller minimizes his/her payoff in the core; and $(\underline{u}, \bar{v})$ the sellers-optimal core allocation, that is, the core allocation in which each seller maximizes his/her payoff while each buyer minimizes his/her payoff. Assume the optimal matching is in the main diagonal, i.e. $b_{11} + b_{22} \geq b_{12} + b_{21}$ and $b_{22} \geq \max\{b_{12}, b_{21}\}$. Then, for each $0 \leq \eta \leq b_{11}$, there exists a core element (u, v) of w_B such that $v_1 = \eta$. Indeed, we know from Demange (1982) and Leonard (1983) that the maximum core-payoff of an agent in a two-sided assignment game is his/her marginal contribution. Then, the reader can check that under the above assumption $\bar{u}_1 = \bar{v}_1 = b_{11}$ and $\underline{u}_1 = b_{11} - \bar{v}_1 = 0$.

Similarly, given a 2×2 assignment game, if it holds $b_{11} + b_{22} \geq b_{12} + b_{21}$ and $b_{11} \geq \max\{b_{12}, b_{21}\}$, then for each $0 \leq \eta \leq b_{22}$ there exists a core element (u, v) of w_B such that $u_2 = \eta$.

Next, we show that, for the particular case of 2×2×2 assignment games, the dominant diagonal property is a necessary and sufficient condition for core stability.

PROOF OF PROPOSITION 4:

Proof. The *"only if"* part is proved in Proposition 3. To prove the *"if"* part, assume A has a dominant diagonal and denote by μ the optimal matching on the main diagonal. Take an allocation $\alpha = (x, y, z)$ that is in the μ-principal section but outside the core. Let us see that α is dominated by some core allocation. Since it is in the μ-principal section, it satisfies the following conditions:

$$x_1 + y_1 + z_1 = a_{111}$$
$$x_2 + y_2 + z_2 = a_{222}.$$

Since (x, y, z) does not belong to the core, assume without loss of generality that $x_2 + y_1 + z_1 < a_{211}$. All other cases are treated similarly. We first look for a core allocation $\beta = (u, v, w)$ that satisfies $u_2 + v_1 + w_1 = a_{211}$ such that β dominates α via coalition $\{2, 1', 1''\}$. This equality, together with the core constraint $u_1 + v_1 + w_1 = a_{111}$ leads to $u_1 = u_2 + a_{111} - a_{211}$. Now, if we had such core allocation β, by substitution in the core constraints, we would get:

(i) $u_2 + v_1 + w_1 = a_{211}$

(ii) $u_2 + v_2 + w_2 = a_{222}$

(iii) $u_2 + v_2 + w_1 \geq a_{121} + a_{211} - a_{111}$

(iv) $u_2 + v_1 + w_1 \geq a_{211}$

(v) $u_2 + v_2 + w_1 \geq a_{221}$

(vi) $u_2 + v_1 + w_2 \geq a_{112} + a_{211} - a_{111}$

(vii) $u_2 + v_2 + w_2 \geq a_{122} + a_{211} - a_{111}$

(viii) $u_2 + v_1 + w_2 \geq a_{212}$.

Note that *(i)* implies *(iv)* and since $\{(1,1',1''),(2,2',2'')\}$ is an optimal matching, *(ii)* implies *(vii)*. By *(iii)* and *(v)* we get $v_2 + w_1 \geq \max\{a_{221} - u_2, a_{121} + a_{211} - a_{111} - u_2, 0\}$ and by *(vi)* and *(viii)* we get $v_1 + w_2 \geq \max\{a_{212} - u_2, a_{112} + a_{211} - a_{111} - u_2, 0\}$. Hence, a core element $\beta = (u, v, w)$ satisfies $u_2 + v_1 + w_1 = a_{211}$ if and only if its projection (v, w) belongs to the core of the 2×2 assignment game defined by matrix B^{u_2}:

$$\begin{pmatrix} a_{211} - u_2 & \max\{a_{212} - u_2, a_{112} + a_{211} - a_{111} - u_2, 0\} \\ \max\{a_{221} - u_2, a_{121} + a_{211} - a_{111} - u_2, 0\} & a_{222} - u_2 \end{pmatrix}.$$

Define $\tilde{u}_2 = x_2 + \varepsilon$ with $0 < \varepsilon < \min\{a_{222} - x_2, a_{211} - x_2 - y_1 - z_1\}$. Notice that this is always possible since $x_2 + y_1 + z_1 < a_{211}$ and because of the dominant diagonal assumption $x_2 < a_{211} \leq a_{222}$. We now consider the matrix $B^{\tilde{u}_2}$.

By the dominant diagonal property and the fact that $\{(1,1',1''),(2,2',2'')\}$ is optimal, we always have

$$\begin{aligned} b_{22}^{\tilde{u}_2} = a_{222} - \tilde{u}_2 \geq \max\Big\{ & \max\{a_{212} - \tilde{u}_2, a_{112} + a_{211} - a_{111} - \tilde{u}_2, 0\}, \\ & \max\{a_{221} - \tilde{u}_2, a_{121} + a_{211} - a_{111} - \tilde{u}_2, 0\}\Big\} \\ = \max\{ & b_{12}^{\tilde{u}_2}, b_{21}^{\tilde{u}_2}\}. \end{aligned} \tag{3}$$

Case 1: $b_{11}^{\tilde{u}_2} + b_{22}^{\tilde{u}_2} \geq b_{12}^{\tilde{u}_2} + b_{21}^{\tilde{u}_2}$. That is,

$$\begin{aligned} a_{211} - \tilde{u}_2 + a_{222} - \tilde{u}_2 \geq & \max\{a_{212} - \tilde{u}_2, a_{112} + a_{211} - a_{111} - \tilde{u}_2, 0\} \\ & + \max\{a_{221} - \tilde{u}_2, a_{121} + a_{211} - a_{111} - \tilde{u}_2, 0\}. \end{aligned}$$

Let us now define

$$\begin{aligned} v_1 &= y_1 + \frac{a_{211} - x_2 - y_1 - z_1 - \varepsilon}{2} > y_1 \geq 0, \\ w_1 &= z_1 + \frac{a_{211} - x_2 - y_1 - z_1 - \varepsilon}{2} > z_1 \geq 0. \end{aligned}$$

Note that $v_1 + w_1 = a_{211} - \tilde{u}_2$ and $v_1 \geq 0, w_1 \geq 0$.

By Remark 10, for all v_1 such that $0 \leq v_1 \leq a_{211} - \tilde{u}_2$ there exists a core allocation $\gamma = (\tilde{v}_1, \tilde{v}_2; \tilde{w}_1, \tilde{w}_2)$ of $B^{\tilde{u}_2}$ with $\tilde{v}_1 = v_1$. Notice that such a core allocation γ satisfies the constraint $\tilde{v}_2 + \tilde{w}_2 = a_{222} - \tilde{u}_2$ since by assumption of Case 1, $\{(1,1'),(2,2')\}$ is optimal for $B^{\tilde{u}_2}$. Then, by completion with $\tilde{u}_1 = \tilde{u}_2 + a_{111} - a_{211}$, we obtain a core allocation, $\beta = (\tilde{u}_1, \tilde{u}_2; \tilde{v}_1, \tilde{v}_2; \tilde{w}_1, \tilde{w}_2)$, of the three-sided assignment game such that $\beta \text{ dom }_{\{2,1,1\}} \alpha$.

Case 2: $b_{12}^{\tilde{u}_2} + b_{21}^{\tilde{u}_2} > b_{11}^{\tilde{u}_2} + b_{22}^{\tilde{u}_2}$.
Since $b_{22}^{\tilde{u}_2} \geq \max\{b_{12}^{\tilde{u}_2}, b_{21}^{\tilde{u}_2}\}$, it holds in this case that $b_{11}^{\tilde{u}_2} < b_{12}^{\tilde{u}_2}$ and $b_{11}^{\tilde{u}_2} < b_{21}^{\tilde{u}_2}$.

To sum up,

$$\begin{aligned}
\max\{a_{212}-\tilde{u}_2, a_{112}+a_{211}-a_{111}-\tilde{u}_2, 0\} &> a_{211}-\tilde{u}_2,\\
\max\{a_{221}-\tilde{u}_2, a_{121}+a_{211}-a_{111}-\tilde{u}_2, 0\} &> a_{211}-\tilde{u}_2,\\
a_{211}-\tilde{u}_2+a_{222}-\tilde{u}_2 < \max\{a_{212}-\tilde{u}_2, &a_{112}+a_{211}-a_{111}-\tilde{u}_2, 0\}\\
+\max\{a_{221}-\tilde{u}_2, &a_{121}+a_{211}-a_{111}-\tilde{u}_2, 0\}. \qquad (4)
\end{aligned}$$

Note that, taking into account the dominant diagonal property, this implies

$$\begin{aligned}
\max\{a_{212}-\tilde{u}_2, a_{112}+a_{211}-a_{111}-\tilde{u}_2, 0\} &= a_{212}-\tilde{u}_2\\
\max\{a_{221}-\tilde{u}_2, a_{121}+a_{211}-a_{111}-\tilde{u}_2, 0\} &= a_{221}-\tilde{u}_2. \qquad (5)
\end{aligned}$$

Then, by (4) and (5), $a_{211}-\tilde{u}_2+a_{222}-\tilde{u}_2 < a_{212}-\tilde{u}_2+a_{221}-\tilde{u}_2$ which is equivalent to $a_{211}+a_{222} < a_{212}+a_{221}$. Hence, $(x_2+y_1+z_2)+(x_2+y_2+z_1) = (x_2+y_1+z_1)+(x_2+y_2+z_2) < a_{211}+a_{222} < a_{212}+a_{221}$. This means that either $x_2+y_1+z_2 < a_{212}$ or $x_2+y_2+z_1 < a_{221}$.

Case 2.1: $x_2+y_1+z_2 < a_{212}$.

We now look for a core allocation $\beta=(u,v,w)$ of w_A such that β dominates α via $\{2,1',2''\}$, and hence $u_2+v_1+w_2 = a_{212}$. Together with the core constraint $u_2+v_2+w_2 = a_{222}$, we get $v_2 = v_1+(a_{222}-a_{212})$.

If we had such core allocation β, by substitution in the core constraints, we would get

(i) $u_1+v_1+w_1 = a_{111}$

(ii) $u_2+v_1+w_2 = a_{222}+a_{212}-a_{222} = a_{212}$

(iii) $u_1+v_1+w_1 \geq a_{121}+a_{212}-a_{222}$

(iv) $u_2+v_1+w_1 \geq a_{211}$

(v) $u_2+v_1+w_1 \geq a_{221}+a_{212}-a_{222}$

(vi) $u_1+v_1+w_2 \geq a_{112}$

(vii) $u_1+v_1+w_2 \geq a_{122}+a_{212}-a_{222}$

(viii) $u_2+v_1+w_2 \geq a_{212}$.

Note that from the fact that $\{(1,1',1''),(2,2',2'')\}$ is optimal for A and the dominant diagonal property, *(i)* implies *(iii)* and *(ii)* implies *(viii)* . By *(vi)* and *(vii)* we get $u_1+w_2 \geq \max\{a_{112}-v_1, a_{122}+a_{212}-a_{222}-v_1, 0\}$ and by *(iv)* and *(v)* we get $u_2+w_1 \geq \max\{a_{211}-v_1, a_{221}+a_{212}-a_{222}-v_1, 0\}$. Hence $\beta=(u,v,w)\in C(w_A)$ satisfies $u_2+v_1+w_2 = a_{212}$ if and only if its projection $(u,w)=(u_1,u_2;w_1,w_2)$ belongs to the core of the 2×2 assignment game B^{v_1}

$$\begin{pmatrix} a_{111}-v_1 & \max\{a_{112}-v_1, a_{122}+a_{212}-a_{222}-v_1, 0\}\\ \max\{a_{211}-v_1, a_{221}+a_{212}-a_{222}-v_1, 0\} & a_{212}-v_1 \end{pmatrix}.$$

Let us now take $\tilde{v}_1 = y_1 + \varepsilon$ where $0 < \varepsilon < \min\{a_{111} - y_1, a_{212} - x_2 - y_1 - z_2\}$. Notice this is always possible since $0 \leq y_1 < a_{212} \leq a_{111}$. Consider now $B^{\tilde{v}_1}$. Note that

$$
\begin{aligned}
b_{11}^{\tilde{v}_1} = a_{111} - \tilde{v}_1 \geq \max \Big\{ & \max\{a_{112} - \tilde{v}_1, a_{122} + a_{212} - a_{222} - \tilde{v}_1, 0\}, \\
& \max\{a_{211} - \tilde{v}_1, a_{221} + a_{212} - a_{222} - \tilde{v}_1, 0\} \Big\} \qquad (6) \\
= \max\{ & b_{12}^{\tilde{v}_1}, b_{21}^{\tilde{v}_1}\}.
\end{aligned}
$$

From $a_{211} + a_{222} < a_{212} + a_{221}$ and $a_{222} \geq a_{221}$ we know that $a_{211} < a_{212}$. Together with (6) this implies that $a_{111} - \tilde{v}_1 + a_{212} - \tilde{v}_1 \geq \max\{a_{112} - \tilde{v}_1, a_{122} + a_{212} - a_{222} - \tilde{v}_1, 0\} + \max\{a_{211} - \tilde{v}_1, a_{221} + a_{212} - a_{222} - \tilde{v}_1, 0\}$, that is $b_{11}^{\tilde{v}_1} + b_{22}^{\tilde{v}_1} \geq b_{12}^{\tilde{v}_1} + b_{21}^{\tilde{v}_1}$.

Let us define

$$
\begin{aligned}
u_2 &= x_2 + \frac{a_{212} - x_2 - y_1 - z_2 - \varepsilon}{2} > x_2 \geq 0, \\
w_2 &= z_2 + \frac{a_{212} - x_2 - y_1 - z_2 - \varepsilon}{2} > z_2 \geq 0.
\end{aligned}
$$

Note that $u_2 + w_2 = a_{212} - \tilde{v}_1$ and $u_2 > 0, w_2 > 0$.

By Remark 10, there exists a core allocation γ of $B^{\tilde{v}_1}$ with $\tilde{u}_2 = u_2$. Such a core allocation γ satisfies the constraint $\tilde{u}_1 + \tilde{w}_1 = a_{111} - \tilde{v}_1$. Then, by completion with $\tilde{v}_2 = \tilde{v}_1 + a_{222} - a_{212}$, we obtain a core allocation of the three-sided assignment game, $(\tilde{u}_1, \tilde{u}_2; \tilde{v}_1, \tilde{v}_2; \tilde{w}_1, \tilde{w}_2)$, such that β dom $_{\{2,1,2\}}\alpha$.

Case 2.2: $x_2 + y_2 + z_1 < a_{221}$.
We now look for a core allocation $\beta = (u, v, w)$ of w_A such that β dominates α via $\{2, 2', 1''\}$ and $u_2 + v_2 + w_1 = a_{221}$. Together with the core constraint $u_2 + v_2 + w_2 = a_{222}$, we get $w_2 = w_1 + (a_{222} - a_{221})$.

If we had such a core allocation β, by substitution in the core constraints we would obtain

(i) $u_1 + v_1 + w_1 = a_{111}$

(ii) $u_2 + v_2 + w_1 = a_{222} + a_{221} - a_{222} = a_{221}$

(iii) $u_1 + v_1 + w_1 \geq a_{112} + a_{221} - a_{222}$

(iv) $u_1 + v_2 + w_1 \geq a_{121}$

(v) $u_1 + v_2 + w_1 \geq a_{122} + a_{221} - a_{222}$

(vi) $u_2 + v_1 + w_1 \geq a_{211}$

(vii) $u_2 + v_1 + w_1 \geq a_{212} + a_{221} - a_{222}$

(viii) $u_2 + v_2 + w_1 \geq a_{221}$.

Note that because of the dominant diagonal property and the fact that the matching $\{(1, 1', 1''), (2, 2', 2'')\}$ is optimal for A, we have *(i)* implies *(iii)* and *(ii)* implies *(viii)*. By *(iv)* and *(v)* we get $u_1 + v_2 \geq \max\{a_{121} - w_1, a_{122} + a_{221} - a_{222} - w_1, 0\}$ and by *(vi)* and *(vii)* we get $u_2 + v_1 \geq \max\{a_{211} - w_1, a_{212} + a_{221} - a_{222} - w_1, 0\}$. Hence, a core element $\beta = (u, v, w)$ satisfies $u_2 + v_2 + w_1 = a_{221}$ if and only if its projection $(u, v) = (u_1, u_2; v_1, v_2)$ belongs to the core of the 2×2 assignment game B^{w_1}:

$$\begin{pmatrix} a_{111} - w_1 & \max\{a_{121} - w_1, a_{122} + a_{221} - a_{222} - w_1, 0\} \\ \max\{a_{211} - w_1, a_{212} + a_{221} - a_{222} - w_1, 0\} & a_{221} - w_1 \end{pmatrix}.$$

Let us now take $\tilde{w}_1 = z_1 + \varepsilon$ where $0 < \varepsilon < \min\{a_{111} - z_1, a_{221} - x_2 - y_2 - z_1\}$. Notice that this is always possible since $0 \leq z_1 < a_{221} \leq a_{111}$. Consider now $B^{\tilde{w}_1}$. Then,

$$\begin{aligned} b_{11}^{\tilde{w}_1} = a_{111} - \tilde{w}_1 \geq \max \Big\{ & \max\{a_{121} - \tilde{w}_1, a_{122} + a_{212} - a_{222} - \tilde{w}_1, 0\}, \\ & \max\{a_{211} - \tilde{w}_1, a_{212} + a_{221} - a_{222} - \tilde{w}_1, 0\} \Big\} \\ = \max\{ & b_{12}^{\tilde{w}_1}, b_{21}^{\tilde{w}_1}\}. \end{aligned} \tag{7}$$

Now, from $a_{211} + a_{222} < a_{212} + a_{221}$ and $a_{222} \geq a_{212}$ we get $a_{221} > a_{211}$, and together with (7) this implies $a_{111} - \tilde{w}_1 + a_{221} - \tilde{w}_2 \geq \max\{a_{121} - \tilde{w}_1, a_{122} + a_{212} - a_{222} - \tilde{w}_1, 0\} + \max\{a_{211} - \tilde{w}_1, a_{212} - \tilde{w}_1 + a_{221} - a_{222}, 0\}$, that is $b_{11}^{\tilde{w}_1} + b_{22}^{\tilde{w}_1} \geq b_{12}^{\tilde{w}_1} + b_{21}^{\tilde{w}_1}$.

Let us define

$$u_2 = x_2 + \frac{a_{221} - x_2 - y_2 - z_1 - \varepsilon}{2} > x_2 \geq 0,$$
$$v_2 = y_2 + \frac{a_{221} - x_2 - y_2 - z_1 - \varepsilon}{2} > y_2 \geq 0.$$

Note that $u_2 + v_2 = a_{221} - \tilde{w}_1$ and $u_2 > 0, v_2 > 0$.

By Remark 10, there exists a core allocation $\gamma = (u_1, u_2; v_1, v_2)$ of $B^{\tilde{w}_1}$ with $\tilde{u}_2 = u_2$. Such a core allocation γ satisfies the constraint $\tilde{u}_1 + \tilde{v}_1 = a_{111} - \tilde{w}_1$. Then, by completion with $\tilde{w}_2 = \tilde{w}_1 + a_{222} - a_{221}$, we obtain a core allocation of the three-sided assignment game $\beta = (\tilde{u}_1, \tilde{u}_2; \tilde{v}_1, \tilde{v}_2; \tilde{w}_1, \tilde{w}_2)$ such that β dom $_{\{2,2,1\}}\alpha$. □

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