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On the Shapley value of liability games

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ABSTRACT

In a liability problem, the asset value of an insolvent firm must be distributed among the creditors and the firm itself, when the firm has some freedom in negotiating with the creditors. We model the negotiations using cooperative game theory and analyze the Shapley value to resolve such liability problems. We establish three main monotonicity properties of the Shapley value. First, creditors can only benefit from the increase in their claims or of the asset value. Second, the firm can only benefit from the increase of a claim but can end up with more or with less if the asset value increases, depending on the configuration of small and large liabilities. Third, creditors with larger claims benefit more from the increase of the asset value. Even though liability games are constant-sum games and we show that the Shapley value can be calculated directly from a liability problem, we prove that calculating the Shapley payoff to the firm is NP-hard.

JEL codes: C71, C78

Keywords: Game theory, Shapley value, constant-sum game, liability game, insolvency

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A Shapley-érték a tartozásos játékokban

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ÖSSZEFOGLALÓ

Tartozásos probléma esetén a fizetésképtelen vállalat eszközeit el kell osztani a hitelezők és a vállalat között, miközben a vállalkozásnak bizonyos szabadsága van a hitelezőkkel folytatott tárgyalások során. A tárgyalásokat kooperatív játékokkal modellezzük, és elemezzük a Shapley-értéket az ilyen tartozásos problémák megoldása érdekében. Belátjuk a Shapley-érték három fő monotonitási tulajdonságát. Először is, a hitelezők mindig jobban járnak, ha nő a követelésük vagy az eszközérték. Másodszor, a vállalat egy tartozás növekedéséből csak profitálhat, de jobban vagy rosszabbul is járhat, ha az eszközérték növekszik, a kis és nagy tartozások összetételétől függően. Harmadszor, a nagyobb követelésekkel rendelkező hitelezők jobban részesülnek az eszközérték növekedéséből. Annak ellenére, hogy a tartozásos játékok konstans összegű játékok, és megmutatjuk, hogy a Shapley-érték közvetlenül kiszámolható egy tartozásos probléma adataiból, a kapcsolódó kooperatív játék generálása nélkül is, bizonyítjuk, hogy a vállalat Shapley-értékének kiszámítása NP-nehéz.

JEL: C71, C78

Kulcsszavak: Játékelmélet, Shapley-érték, konstans összegű játék, tartozásos játék, fizetésképtelenség

On the Shapley value of liability games*

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Abstract

In a liability problem, the asset value of an insolvent firm must be distributed among the creditors and the firm itself, when the firm has some freedom in negotiating with the creditors. We model the negotiations using cooperative game theory and analyze the Shapley value to resolve such liability problems. We establish three main monotonicity properties of the Shapley value. First, creditors can only benefit from the increase in their claims or of the asset value. Second, the firm can only benefit from the increase of a claim but can end up with more or with less if the asset value increases, depending on the configuration of small and large liabilities. Third, creditors with larger claims benefit more from the increase of the asset value. Even though liability games are constant-sum games and we show that the Shapley value can be calculated directly from a liability problem, we prove that calculating the Shapley payoff to the firm is NP-hard.

Keywords: Game theory · Shapley value · constant-sum game · liability game · insolvency

JEL Classification: C71 · C78

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1 Introduction

An insolvent firm (country, state, individual, etc.) with some asset value has liabilities towards a group of creditors. Compared to standard bankruptcy games as studied in the game-theoretical literature (see O'Neill (1982) for a seminal contribution and Thomson (2013), and Thomson (2015) for recent surveys) Csóka and Herings (2019) introduced liability problems, by modeling the firm as an explicit player. A liability problem is given by the asset value of the firm to be allocated and the claims of the creditors.

Instead of directly using the values given in a liability problem, Csóka and Herings (2019) defined liability games to indirectly allocate the asset value using a solution concept from cooperative game theory. The worth of a coalition in a liability game is defined as follows. Given a coalition and its complement, the firm first makes payments to the coalition it belongs to, up to the value of the liabilities in the firm's coalition and the asset value of the firm, and then (if possible) pays to the complementary coalition. They remarked that liability games are superadditive: there is no loss of merging disjoint coalitions. Moreover, they proved that the core of a liability game is empty and analyzed one of the two most popular solution concepts, the nucleolus (Schmeidler, 1969).

In this paper, we investigate the Shapley value (Shapley, 1953) of liability games. The numerous applications of the Shapley value include aircraft landing fees (Littlechild and Owen, 1973; Dubey, 1982), minimal cost spanning trees (Bergantinos and Lorenzo-Freire, 2008), a combinatorial structure called augmenting system (Bilbao and Ordóñez, 2009), directed graph games (Khmelnitskaya, Selçuk, and Talman, 2016), risk capital allocation (Balog, Bátyi, Csóka, and Pintér, 2017), and for environmental costs in supply chains (Ciardiello, Genovese, and Simpson, 2018) among others.

We show that the Shapley value can also be used as an allocation rule, that is, it allocates the asset value non-negatively among the creditors and the firm in such a way that no creditor gets more than his liability. We establish lower and upper bounds for the Shapley payments. Moreover, we show that (i) creditors can only benefit from the increase in their claims or of the asset value; (ii) the firm can only benefit from the increase of a claim but can end up with more or with less if the asset value increases, depending on the configuration of small and large liabilities; (iii) creditors with larger claims benefit more from the increase of the asset value. In most cases, we even establish sharp upper bounds for the changes in the payments.

It is easy to check that in liability games, for one or two creditors (that is, for two or three players), the Shapley value coincides with the nucleolus. However, for three or more creditors, they give different payoffs in generic examples. Csóka and Herings (2019) showed that at the nucleolus of a liability game, the firm gets a positive payment, which is at most half of the asset value. We show that at the Shapley value, there are cases when the firm can keep almost the whole asset value. Csóka and Herings (2019) also showed that at the nucleolus, creditors with higher liabilities receive higher payments, but they also get higher debt forgiveness (defined as the difference between the liability and the received payments), a result we also have for the Shapley value. They also provided conditions under which the nucleolus coincides with a generalized proportional rule, where the firm gets a positive amount, and the rest is allocated in proportional to the liabilities.

Csóka and Herings (2019) noted that in a liability game, the worth of a coalition plus the worth of the complementary coalition is always equal to the asset value, that is, a liability game is a constant-sum game (Von Neumann and Morgenstern, 1944). Originally, Von Neumann and Morgenstern (1944) analyzed strategic non-cooperative games, where a coalition and the complementary coalition play a constant-sum game. They discussed constant-sum simple games with winning or losing coalitions, where the worth of any coalition can be either zero or one. A prominent application is (weighted) majority voting games, where the worth of the grand coalition is one, and if a coalition is winning, then its complementary coalition is losing. Constant-sum games also play a role in games modeling Bitcoin mining pools (Lewenberg, Bachrach, Sompolinsky, Zohar, and Rosenschein, 2015). For a recent generalization to alpha-constant-sum games, see Wang, van den Brink, Sun, Xu, and Zou (2019). A related new concept is called games of threats (Kohlberg and Neyman, 2018), where the constant-sum is zero, but the value of the empty coalition is not always zero. For more details on the value theory of strategic games, see Cai, Candogan, Daskalakis, and Papadimitriou (2016).

Since constant-sum games are exciting on their own, we first study the Shapley value for constant-sum games in general. We propose a basis for the linear vector space of constant-sum games that provides a specialized formula for the Shapley payoff to a player in a constant-sum game. It turns out that some of those general results are very handy for liability games. We obtain a simple computational scheme by which the Shapley value of a liability game is derived directly from the liability problem, that is, from the asset value and the liabilities.

In general, computing the Shapley value based on its definition is practically impossible for large games. Computing the Shapley value in weighted majority games is #P-complete (Deng and Papadimitriou, 1994) and one has to rely on its estimation. Estimation techniques were introduced by Castro, Gómez, and Tejada (2009) and Castro, Gómez, Molina, and Tejada (2017). However, for special classes of games, the Shapley value can be calculated in a polynomial manner (Megiddo, 1978; Granot, Kuipers, and

Chopra, 2002; Castro, Gómez, and Tejada, 2008). We show that in liability games, calculating the Shapley value of the insolvent firm is NP-hard. Thus even though the Shapley value can be calculated directly from the liability problem, its application to large liability problems could become computationally laborious.

The paper is organized as follows. In Section 2, we consider general constant-sum games. In Section 3, we introduce liability games, show that the Shapley value can be used as an allocation rule, and provide two examples. In Section 4, we prove various properties of the Shapley allocation rule. Section 5, we show that calculating the Shapley value of the firm is NP-hard.

2 The Shapley value of constant-sum games

A *transferable utility cooperative game* (N, v) is a pair where N is a non-empty, finite set of *players* and $v : 2^N \to \mathbb{R}$ is a *coalitional function* satisfying $v(\emptyset) = 0$. The number $v(S)$ is regarded as the worth of the coalition $S \subseteq N$. We identify the game with its coalitional function since the player set N is fixed throughout the paper. The game (N, v) is called *0-normalized* if $v(\{i\}) = 0$ for every $i \in N$; *superadditive* if $S \cap T = \emptyset$ implies $v(S) + v(T) \leq v(S \cup T)$ for every two coalitions $S, T \subseteq N$. The game (N, v) is *constant-sum* if $v(S) + v(N \setminus S) = v(N)$ for every coalition $S \subseteq N$.

Given a game (N, v), a *payoff allocation* $x \in \mathbb{R}^N$ represents the payoffs to the players. The total payoff to coalition $S \subseteq N$ is denoted by $x(S) = \sum_{i \in S} x_i$ if $S \neq \emptyset$ and $x(\emptyset) = 0$. In a game v, we say the payoff allocation x is *efficient*, if $x(N) = v(N)$; *individually rational*, if $x_i = x(\{i\}) \geq v(\{i\})$ for all $i \in N$; *coalitionally rational*, if $x(S) \geq v(S)$ for all $S \subseteq N$. The set of *preimputations*, $I^*(v)$, consists of the efficient payoff vectors, the set of *imputations*, $I(v)$, consists of the individually rational preimputations, and the *core*, $C(v)$, is the set of coalitionally rational (pre)imputations. We call a game *balanced* if its core is non-empty.

We denote the set of all cooperative games on a fixed player set N by $\mathcal{G}^N$. It is well-known that $\mathcal{G}^N$ is a linear vector space of dimension $2^n - 1$ where $n = |N|$. A *value* on $\mathcal{G}^N$ is a map $f : \mathcal{G}^N \to \mathbb{R}^N$, which assigns to every game v on N a vector $f(v)$ with components $f_i(v)$ for all $i \in N$. We say that value f satisfies

- *linearity*: if $f(\alpha v + \beta w) = \alpha f(v) + \beta f(w)$ holds for all $\alpha, \beta \in \mathbb{R}$ and $v, w \in \mathcal{G}^N$.
- *efficiency*: if $\sum_{j \in N} f_j(v) = v(N)$ holds for all $v \in \mathcal{G}^N$.
- *equal treatment property*: if $j, k \in N$ are *symmetric* players in game $v \in \mathcal{G}^N$, that is if $v(S \cup j) = v(S \cup k)$ $\forall S \subseteq N \setminus \{j, k\}$, then $f_j(v) = f_k(v)$.

- *dummy player property*: if $j \in N$ is a *dummy* player in game $v \in \mathcal{G}^N$, that is if $v(S \cup j) - v(S) = v(j)\ \forall S \subseteq N \setminus j$, then $f_j(v) = v(j)$.

The best known and most frequently used value was introduced and characterized by a few appealing properties by Lloyd Shapley.

Theorem 1. *(Shapley, 1953) The value $\phi : \mathcal{G}^N \to \mathbb{R}^N$ defined by*

$$\phi_i(v) = \sum_{S \subseteq N \setminus i} \gamma_N(S)[v(S \cup i) - v(S)] \qquad (i \in N) \tag{1}$$

where $\gamma_N(S) = \dfrac{s!(n-1-s)!}{n!} = \dfrac{1}{n\binom{n-1}{s}}$ and $s = |S|, n = |N|$, is the only value on $\mathcal{G}^N$ that satisfies linearity, efficiency, the equal treatment property, and the dummy player property.

The Shapley value can also be axiomatized using a different set of axioms, see Young (1985) for the axioms, Pintér (2015) for different classes of games and in particular Khmelnitskaya (2003) for constant-sum games.

It is well-known that the weight coefficients $\{\gamma_N(S)\}_{S \subseteq N \setminus i}$ form a probability distribution, we call it the *Shapley distribution*, on the family $2^{N \setminus i}$ of coalitions that does not contain player i. Therefore, $\phi_i(v)$ is the expected marginal contribution of player i in v to coalitions not containing i, when the random formation of such coalitions is described by the Shapley distribution. Since $\gamma_N(S)$ depends only on the cardinalities $n = |N|$ and $s = |S|$ of the two coalitions, we also write $\gamma_n(s)$ when more convenient.

Next, we derive a specialized formula for the Shapley value of constant-sum games. We denote the set of all constant-sum games on fixed player set N by $\mathcal{G}^N_{CS}$.

Proposition 2. *The Shapley value of constant-sum game $v \in \mathcal{G}^N_{CS}$ is*

$$\phi_i(v) = -v(N) + 2 \sum_{S \subseteq N \setminus i} \gamma_N(S) v(S \cup i) \qquad (i \in N). \tag{2}$$

Proof. Let v be a constant-sum game and $i \in N$ be fixed. For $S \subseteq N \setminus i$, we have $v(S) = v(N) - v(N \setminus S) = v(N) - v((N \setminus i \setminus S) \cup i)$. If we substitute this in the general formula (1), we get $\phi_i(v) = \sum_{S \subseteq N \setminus i} \gamma_N(S)[v(S \cup i) + v((N \setminus i \setminus S) \cup i) - v(N)]$. Since $\gamma_n(s) = \frac{1}{n\binom{n-1}{s}} = \frac{1}{n\binom{n-1}{n-1-s}} = \gamma_n(n-1-s)$ and $N \setminus i \setminus S \subseteq N \setminus i$, each coalition value of type $v(T \cup i)$ for $T \subseteq N \setminus i$ appears twice and is weighted by the same coefficient in the sum. Taking out the constant term $-v(N)$ from the summation, we get formula (2). $\square$

Notice that in constant-sum games, the Shapley payoff to a player depends on the values of coalitions the player belongs to, no need to compute his marginal contributions.

Next, we investigate how the Shapley value of constant-sum games can be computed based on its linearity. It is easily seen that any linear combination of constant-sum games is also a constant-sum game. Thus $\mathcal{G}^N_{CS}$ is a linear subspace of $\mathcal{G}^N$. It is well-known that additive games are the only balanced constant-sum games, so the standard approach of decomposing a game as a linear combination of unanimity games cannot be followed for $\mathcal{G}^N_{CS}$. Only the additive unanimity games, that is, the dictator games $u_{\{i\}}$ $(i \in N)$, could be part of a basis for $\mathcal{G}^N_{CS}$, but they are sufficient to span only the n-dimensional linear subspace of $\mathcal{G}^N_{CS}$ consisting of the additive constant-sum games.

Foreshadowing the application of these game-theoretic results to a special type of constant-sum games induced by liability problems with an insolvent firm, we arbitrarily choose a player (the insolvent firm) and denote him by $0 \in N$. The set of the $n-1$ other players is denoted by $C = N \setminus \{0\}$. Given this fixed "highlighted" player, the family of all coalitions is decomposed in two parts of equal size: the 2^{n-1} "partner" coalitions containing 0 and the 2^{n-1} "complement" coalitions. Let $\mathcal{P}_0 = \{S \subseteq N : 0 \in S\}$ denote the family of partner coalitions of 0, and $\mathcal{C}_0 = \{S \subseteq N : 0 \notin S\}$ denote the family coalitions not containing 0. Obviously, $S \in \mathcal{P}_0$ if and only if $N \setminus S \in \mathcal{C}_0$, In particular, $N \in \mathcal{P}_0$ and $\emptyset \in \mathcal{C}_0$, also $\{0\} \in \mathcal{P}_0$ and $C \in \mathcal{C}_0$.

In a constant-sum game $v \in \mathcal{G}^N_{CS}$, we have $v(N \setminus S) = v(N) - v(S)$ for all $S \in \mathcal{P}_0$, thus the values of the partner coalitions $v(S)$ $(S \in \mathcal{P}_0)$ suffice to fully determine v. It follows that the dimension of $\mathcal{G}^N_{CS}$ is at most $2^{n-1} = |\mathcal{P}_0|$. Next, we show that, in fact, equality holds. We present 2^{n-1} linearly independent "elementary" constant-sum games, which form a very "convenient" basis of $\mathcal{G}^N_{CS}$, the scalar coefficients in the (unique) linear decompositions are simply the coalitional values.

We define for $0 \in R \subsetneq N$ the constant-sum game $d^R \in \mathcal{G}^N_{CS}$ for all $S \subseteq N$ by

$$d^R(S) = \begin{cases} 1, & \text{if } S = R, \\ -1, & \text{if } S = N \setminus R, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

For $R = N$, the constant-sum game $d^N \in \mathcal{G}^N_{CS}$ is defined for all $S \subseteq N$ as

$$d^N(S) = \begin{cases} 1, & \text{if } S = N \text{ or } 0 \notin S \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

It is easily checked that $d^R(\emptyset) = 0$ and d^R is indeed constant-sum for all $R \in \mathcal{P}_0$. Moreover, $d^N(N) = 1$ but $d^R(N) = 0$ for all $N \neq R \in \mathcal{P}_0$. Notice that for all $R, S \in \mathcal{P}_0$, we have $d^R(S) = 1$ if and only if $R = S$, but $d^R(S) = 0$ otherwise. It follows that the $2^{n-1} = |\mathcal{P}_0|$ games d^R ($R \in \mathcal{P}_0$) are linearly independent in $\mathcal{G}^N_{CS}$.

We summarize the above discussion in the following proposition.

Proposition 3. *The games $d^R \in \mathcal{G}^N_{CS}$ ($R \in \mathcal{P}_0$) form a basis of $\mathcal{G}^N_{CS}$, henceforth $\dim(\mathcal{G}^N_{CS}) = 2^{n-1}$. Moreover, $v(S) = \sum_{R\in\mathcal{P}_0} v(R) \cdot d^R(S)$ for all $S \subseteq N$ and $v \in \mathcal{G}^N_{CS}$. Consequently, by linearity of the Shapley value, $\phi(v) = \sum_{R\in\mathcal{P}_0} v(R) \cdot \phi(d^R)$.*

The following example illustrates this proposition and foreshadows the subsequent general discussion. For the sake of compactness, coalitions will be described without braces and separating commas but overlined: for example, $\overline{0jk}$ means coalition $\{0, j, k\}$. Its value is shorthanded as $v_{\overline{0jk}} = v(\overline{0jk})$.

Example 4 ($n = 3$, $N = \overline{0} \cup \overline{12}$)**.**

	$\mathcal{P}_0$				$\mathcal{C}_0$			
S	$\overline{0}$	$\overline{01}$	$\overline{02}$	N	$\varnothing$	$\overline{1}$	$\overline{2}$	$\overline{12}$
$v(S)$	$v_{\overline{0}}$	$v_{\overline{01}}$	$v_{\overline{02}}$	v_N	0	$v_N - v_{\overline{02}}$	$v_N - v_{\overline{01}}$	$v_N - v_{\overline{0}}$
$d^{\overline{0}}(S)$	1	0	0	0	0	0	0	-1
$d^{\overline{01}}(S)$	0	1	0	0	0	0	-1	0
$d^{\overline{02}}(S)$	0	0	1	0	0	-1	0	0
$d^N(S)$	0	0	0	1	0	1	1	1

Trivially, $v(S) = v_{\overline{0}} \cdot d^{\overline{0}}(S) + v_{\overline{01}} \cdot d^{\overline{01}}(S) + v_{\overline{02}} \cdot d^{\overline{02}}(S) + v_N \cdot d^N(S)$ for all $S \subseteq N$ and $v \in \mathcal{G}^N_{CS}$. We get that the Shapley value of any 3-player constant-sum game v (with distinguished player 0) can be computed as the linear combination of the Shapley values of the above four constant-sum basis games: $\phi(v) = v_{\overline{0}} \cdot \phi(d^{\overline{0}}) + v_{\overline{01}} \cdot \phi(d^{\overline{01}}) + v_{\overline{02}} \cdot \phi(d^{\overline{02}}) + v_N \cdot \phi(d^N)$.

By formula (2), the Shapley payoffs to our special player 0 in the basis games: $\phi_0(d^{\overline{0}}) = 2\gamma_3(0) = 2/3$, $\phi_0(d^{\overline{01}}) = 2\gamma_3(1) = 2/6$, $\phi_0(d^{\overline{02}}) = 2\gamma_3(1) = 2/6$, and $\phi_0(d^N) = -1 + 2\gamma_3(2) = -1/3$.

The payoffs to players 1 and 2 are then easily obtained from the efficiency and equal treatment property of the Shapley value. In $d^{\overline{0}}$ players 1 and 2 are symmetric, in $d^{\overline{01}}$ players 0 and 1 are symmetric, in $d^{\overline{02}}$ players 0 and 2 are symmetric, finally in d^N again the complement players 1 and 2 are symmetric. By simple arithmetic, we get the sharing system (5).

$$
\begin{array}{c||c|cc||c||}
 & \phi_0 & \phi_1 & \phi_2 & v \in \mathcal{G}_{CS}^{\overline{012}} \\
\hline\hline
d^{\overline{0}} & 2/3 & -1/3 & -1/3 & \cdot v_{\overline{0}} \\
\hline
d^{\overline{01}} & 1/3 & 1/3 & -2/3 & \cdot v_{\overline{01}} \\
\hline
d^{\overline{02}} & 1/3 & -2/3 & 1/3 & \cdot v_{\overline{02}} \\
\hline
d^{N} & -1/3 & 2/3 & 2/3 & \cdot v_{N} \\
\hline
\end{array}
\tag{5}
$$

The Shapley payoffs are easily computed from sharing system (5) for any 3-player constant-sum game v with distinguished player 0. We simply take the linear combination of the "partner" coalition values weighted with the "shares" of the given player. In formula,

$$
\phi_0(v) = \frac{2v_{\overline{0}} + v_{\overline{01}} + v_{\overline{02}} - v_N}{3}, \qquad \phi_i(v) = \frac{-v_{\overline{0}} + v_{\overline{0i}} - 2v_{\overline{0j}} + 2v_N}{3} \quad (i \neq j). \tag{6}
$$

The arguments of the above example can be generalized to obtain a similar sharing system in general. Since the basis game values $d^R(S)$ $(R, S \in \mathcal{P}_0)$ form a unit matrix, by formula (2), using $r = |R|$, the Shapley payoffs to our special player 0 in the basis games are

$$
\phi_0(d^R) = \begin{cases} 2\gamma_n(r-1), & \text{if } R \neq N, \\ -1 + 2\gamma_n(n-1,) & \text{if } R = N. \end{cases} \tag{7}
$$

The payoffs to the players in $C = N \setminus \{0\}$ can then be easily obtained from the efficiency and equal treatment property of the Shapley value.

For $R \in \mathcal{P}_0 \setminus \{N\}$, in basis game d^R the players in R are all symmetric, so $\phi_0(d^R) = \phi_i(d^R)$ for all $i \in R$. Similarly, the players in $N \setminus R$ are all symmetric, so $\phi_j(d^R) = \phi_k(d^R)$ for all $j, k \in N \setminus R$. Since $d^R(N) = 0$, efficiency gives $r\phi_0(d^R) + (n-r)\phi_k(d^R) = 0$, where $k \in N \setminus R$. From (7) we easily derive the Shapley payoffs in basis game d^R when $R \neq N$.

$$
\phi_i(d^R) = \begin{cases} 2\gamma_n(r-1), & \text{if } i \in R, \\ -2\gamma_n(r), & \text{if } i \in N \setminus R. \end{cases} \tag{8}
$$

For $R = N$, in basis game d^N all non-distinguished players in C are symmetric, so $\phi_j(d^N) = \phi_k(d^N)$ for all $j, k \in N \setminus \{0\}$. Since $d^N(N) = 1$, efficiency gives $\phi_0(d^N) + (n -$

$1)\phi_k(d^N) = 1$, where $k \neq 0$. From (7), we easily get the Shapley payoffs in d^N as

$$\phi_i(d^N) = \begin{cases} -1 + 2\gamma_n(n-1), & \text{if } i = 0, \\ 2\gamma_n(n-1), & \text{if } i \neq 0. \end{cases} \tag{9}$$

Sharing system (10) schematically summarizes the above formulas. The columns correspond to the partner coalitions of the form $R = \{0\} \cup S$. The rows give the Shapley values of players in the basis constant-sum games, first for our distinguished player 0, second for a generic other player $i \in C$.

$$\begin{array}{c||c|c|cc|c|c||c}
 & s = 0 & \cdots & \multicolumn{2}{c|}{s = |S|} & \cdots & s = n-1 & \\
 & i \notin S & \cdots & i \in S & i \notin S & \cdots & i \in S & \\
\hline
 & \binom{n-2}{0} & \cdots & \binom{n-2}{s-1} & \binom{n-2}{s} & \cdots & \binom{n-2}{n-1} & \\
 & \binom{n-1}{0} & \cdots & \multicolumn{2}{c|}{\binom{n-1}{s}} & \cdots & \binom{n-1}{n-1} & \\
\hline\hline
\phi_0 & 2\gamma_n(0) & \cdots & 2\gamma_n(s) & 2\gamma_n(s) & \cdots & -1 + 2\gamma_n(n-1) & = 1 \\
\hline
\phi_i & -2\gamma_n(1) & \cdots & 2\gamma_n(s) & -2\gamma_n(s+1) & \cdots & 2\gamma_n(n-1) & = 0 \\
\hline\hline
 & = 0 & \cdots & \multicolumn{2}{c|}{= 0} & \cdots & = 1 &
\end{array} \tag{10}$$

Any given player $i \in C$ can either be a partner of player 0 or not. Thus, except when $S = \emptyset$ or $S = C$, among the $\binom{n-1}{s}$ coalitions $S \subseteq C$ of size $1 \leq s \leq n-2$ there are $\binom{n-2}{s-1}$ coalitions which contain i, the remaining $\binom{n-2}{s}$ coalitions do not contain i. The following features of the Shapley sharing system are easily checked.

Proposition 5. *In the Shapley sharing system (10)*

1. *the ϕ_0 row sum $= 1$, every other ϕ_i ($i \in C$) row sum $= 0$;*

2. *the $s = n-1$ column sum $= 1$, every other $0 \leq s \leq n-2$ column sum $= 0$.*

Although in a general constant-sum game distinguishing one arbitrarily picked player served only technical purposes, next, we discuss a special type of constant-sum games where one player is indeed "different" from the other players.

3 Liability games and the Shapley value

We consider a special class of constant-sum games, liability games, introduced by Csóka and Herings (2019).

Let $N = \{0, 1, \ldots, c\}$ denote the set of agents, where agent 0 is a *firm* having a set of *creditors* $C = \{1, \ldots, c\}$ with cardinality $|C| = c \geq 1$. The firm has *asset value* $A \in \mathbb{R}_+$ and *liabilities* $\ell \in \mathbb{R}^C_+$, with $\ell_i \in \mathbb{R}_+$ the liability to creditor $i \in C$. The question is how to allocate the asset value among the creditors and the firm. If the firm is solvent, that is, $\sum_{i \in C} \ell_i \leq A$, then the obvious solution is that every creditor receives its full claim and the firm keeps the rest. Henceforth we only consider the insolvent case, but for ease of presentation, we also allow borderline solvency, that is, $\sum_{i \in C} \ell_i = A$.

Definition 6. A *liability problem* is a pair $(A, \ell) \in \mathbb{R}_+ \times \mathbb{R}^C_+$ such that $\sum_{i \in C} \ell_i \geq A$.

Let $\mathcal{L}^N$ denote the class of liability problems[1] on set of agents $N = \{0\} \cup C$. We seek a liability rule that assigns a unique allocation to each liability problem.

Definition 7 (Csóka and Herings (2019))**.** A *liability rule* is a function $f : \mathcal{L}^N \to \mathbb{R}^N_+$ such that, for every $(A, \ell) \in \mathcal{L}^N$, the payment vector $f = f(A, \ell) \in \mathbb{R}^N$ is an *allocation*, that is a non-negative vector $f \in \mathbb{R}_+ \times \mathbb{R}^C_+$ satisfying *liabilities boundedness*, that is, $f_i \leq \ell_i$ for all $i \in C$, and *efficiency*, that is, $\sum_{i \in N} f_i = A$.

Note that by non-negativity and efficiency, the payments in allocation $f \in \mathbb{R}^N$ fall between the following bounds:

$$0 \leq f_0 \leq A \quad \text{and} \quad 0 \leq f_i \leq \ell^A_i \quad \text{for all } i \in C,$$

where $\ell^A_i = \min\{A, \ell_i\}$ is the *truncated liability* of creditor $i \in C$. Let $\ell^A \in \mathbb{R}^C_+$ denote the vector of liabilities truncated by the asset value.

Given a subset of creditors $S \subseteq C$, we will use the notation $\ell_S = \ell(S) = \sum_{i \in S} \ell_i$ for the total liabilities of S and $\ell^A(S) = \sum_{i \in S} \ell^A_i$ for the total truncated liabilities of S. On the other hand, we will also use the shorthand $\ell^A_S = \min\{A, \ell(S)\} = \min\{A, \ell^A(S)\}$ for the *truncated total (truncated) liabilities* of creditor group $S \subseteq C$. Clearly, $\ell^A_S \leq \ell^A(S)$, and equality holds if and only if $\ell(S) \leq A$.

A liability problem gives rise to a transferable utility cooperative game called liability game (Csóka and Herings, 2019).

Definition 8. Let $(A, \ell) \in \mathcal{L}^N$ be a liability problem. On player set N, the induced *liability game* $v : 2^N \to \mathbb{R}$ is defined by setting, for $S \in 2^N$,

$$v(S) = \begin{cases} \min\{A, \ell(S \setminus \{0\})\} = \ell^A_{S \setminus \{0\}}, & \text{if } 0 \in S, \\ \max\{0, A - \ell(C \setminus S)\}, & \text{if } 0 \notin S. \end{cases}$$

[1] Csóka and Herings (2019) considers a slightly restricted class, when all liabilities are at most as large as the asset value, the asset value is strictly positive, there are at least two creditors and the firm is insolvent.

Note that $v(\emptyset) = 0$, $0 \leq v(S) \leq A$ for all $S \in 2^N$, and $v(N) = A$. Csóka and Herings (2019) notes that liability games are *superadditive*, that is, for all $S, T \in 2^N$, $S \cap T$ implies $v(S) + v(T) \leq v(S \cup T)$; and *constant-sum*, that is, for all $S \in 2^N$, $v(S) + v(N \setminus S) = v(N)$. Due to their superadditivity and nonnegativity, liability games are *monotonic*, that is, for all $S, T \in 2^N$, $S \subset T$ implies $v(S) \leq v(T)$.

We aim to define a liability allocation rule by applying the Shapley value to the induced liability game. This works in practice only if we can compute the Shapley-vector of the liability game directly from the data of the underlying liability problem, that is, from the asset value and the liabilities. The following straightforward observation implies that our indirect approach could only provide a liability rule that ignores excessive parts of the claims. Notice that cutting off the parts of liabilities over the asset value does not make the firm solvent, that is, $\ell(C) \geq A$ implies $\ell^A(C) \geq A$.

Remark 9. Liability problems (A, ℓ) and (A, ℓ^A) induce the same liability game, where ℓ^A denotes the vector of liabilities truncated by the asset value.

It follows that the Shapley rule (or any other allocation rule defined via a single-valued solution of the induced game) is different from rules that allocate (some portion of) the asset value among the creditors proportional to their claims. We will see in Example 11 that the Shapley rule is also different from rules that allocate (some portion of) the asset value among the creditors proportional to their truncated liabilities.

Next, we show that the Shapley value indeed defines a liability rule, that is, the Shapley-vector of the liability game associated with a liability problem is an allocation.

Proposition 10. *Let $(A, \ell) \in \mathcal{L}^N$ be a liability problem and let v be the induced liability game on N. Then the Shapley-vector $\phi(v)$ of v satisfies efficiency, non-negativity, and (truncated) liabilities boundedness.*

Proof. The Shapley value assigns an efficient vector to any TU game, so for any liability game (N, v) we have $\sum_{i \in N} \phi_i(v) = v(N) = A$. The other two properties follow from formula (1), once we show $0 \leq v(S \cup 0) - v(S)$ for all $S \subseteq C = N \setminus 0$ and $0 \leq v(S \cup i) - v(S) \leq \ell_i$ for all $i \in C$, $S \subseteq N \setminus i$.

All marginal contributions of the firm are non-negative. Indeed, $v(S \cup 0) - v(S) = \min\{\ell(S), A\} - \max\{A - \ell(C \setminus S), 0\}$ is obviously non-negative if the second term is zero. If it is positive, that is, $A > \ell(C \setminus S)$, then insolvency of the firm gives $\ell(S) \geq A - \ell(C \setminus S)$, and that, coupled with the obvious $A \geq A - \ell(C \setminus S)$, implies non-negativity of the marginal contribution.

Now let $i \in C$ be a creditor and $S \subseteq N \setminus i$. We have two cases. If $0 \in S$, so $v(S \cup i) - v(S) = \min\{\ell(S \setminus 0) + \ell_i, A\} - \min\{\ell(S \setminus 0), A\}$, then the difference is clearly between 0 and ℓ_i. If $0 \notin S$, so $v(S \cup i) - v(S) = \max\{A - \ell(C \setminus S) + \ell_i, 0\} - \max\{A - \ell(C \setminus S), 0\}$, then again the difference is clearly between 0 and ℓ_i. Thus, we get non-negativity of all marginal contributions for all creditors, as well as liabilities boundedness.

Since non-negativity and efficiency imply $\phi_i \leq A$ for all $i \in N$, including the firm, for creditor $i \in C$ we can sharpen the upper bound to $\phi_i \leq \ell_i^A$. $\square$

Next, we define (truncated) debt forgiveness of a creditor as the difference between the (truncated) liability towards him and the payment he receives. Formally, let $(A, \ell) \in \mathcal{L}^N$ be a liability problem and $x \in \mathbb{R}_+^N$ be an allocation. The *debt forgiveness* of creditor $i \in C$ is given by $\ell_i - x_i$. The *truncated debt forgiveness* by creditor $i \in C$ is given by $\ell_i^A - x_i = \min\{A, \ell_i\} - x_i$.

Example 11. Consider a generic liability problem with two creditors, so $N = \{0, 1, 2\}$ and $A \leq \ell_1 + \ell_2$. The induced liability game v is the following:

S	$\{0\}$	$\{1\}$	$\{2\}$	$\{0,1\}$	$\{0,2\}$	$\{1,2\}$	$\{0,1,2\}$
$v(S)$	0	$A - \ell_2^A$	$A - \ell_1^A$	ℓ_1^A	ℓ_2^A	A	A

The Shapley-payments can be obtained from formulas (6) derived for 3-player constant-sum games. The bounds follow from $A \leq \ell_1 + \ell_2$ implying $A \leq \ell_1^A + \ell_2^A \leq 2A$.

For the firm,

$$0 \leq \phi_0 = \frac{\ell_1^A + \ell_2^A - A}{3} \leq \frac{A}{3}.$$

Clearly both bounds are sharp. Notice that at the Shapley allocation, an insolvent firm ends up with a strictly positive payoff.

For creditor $i \neq j \in C$, since $0 \leq A - \ell_j^A \leq \ell_i^A$,

$$\frac{\ell_i^A}{3} \leq \phi_i = \frac{\ell_i^A - 2\ell_j^A + 2A}{3} = \ell_i^A - 2\phi_0 \leq \ell_i^A.$$

It is easily seen that both bounds are sharp. For the debt forgiveness and for the truncated debt forgiveness of creditor $i \in C$, we immediately get the following sharp bounds:

$$\ell_i - \ell_i^A \leq \ell_i - \phi_i \leq \ell_i - \frac{\ell_i^A}{3}, \qquad 0 \leq \ell_i^A - \phi_i = 2\phi_0 \leq \frac{2\ell_i^A}{3}.$$

Observe that both creditors give the same truncated debt forgiveness ($2\phi_0$) to the firm.

It also follows from the above formulas that if $\ell_i \leq \ell_j$, hence also $\ell_i^A \leq \ell_j^A$, then $\phi_i \leq \phi_j$ and $\ell_i - \phi_i \leq \ell_j - \phi_j$. That is, at the Shapley allocation, the creditor with higher claim gets higher payment, but it also gives an at least as high debt forgiveness.

Example 12. Consider a generic liability problem with three creditors, so $C = \{1, 2, 3\}$, $N = \{0\} \cup C$, and $A \leq \ell_1 + \ell_2 + \ell_3$. We compute the Shapley allocation from sharing system (10) derived for constant-sum games using only values of coalitions, which contain the distinguished player 0, now the firm. For $n = 4$ we get

$S \ni 0$	$\{0\}$	$\{0,1\}$	$\{0,2\}$	$\{0,3\}$	$\{0,1,2\}$	$\{0,1,3\}$	$\{0,2,3\}$	N
$v(S)$	0	ℓ_1^A	ℓ_2^A	ℓ_3^A	ℓ_{12}^A	ℓ_{13}^A	ℓ_{23}^A	A
ϕ_0	$1/2$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$-1/2$
ϕ_1	$-1/6$	$1/6$	$-1/6$	$-1/6$	$1/6$	$1/6$	$-1/2$	$1/2$
ϕ_2	$-1/6$	$-1/6$	$1/6$	$-1/6$	$1/6$	$-1/2$	$1/6$	$1/2$
ϕ_3	$-1/6$	$-1/6$	$-1/6$	$1/6$	$-1/2$	$1/6$	$1/6$	$1/2$

(11)

where $\ell_S^A = \min\{A, \ell(S)\} = \min\{A, \ell^A(S)\}$. The Shapley payments are obtained by multiplying row $[v(S)]$ of the coalition values by row $[\phi_k]$ of the shares for player $k \in N$.

The Shapley payment of the firm is $\phi_0 = \frac{1}{6}\sum_{i \in C}(\ell_i^A + \ell_{jk}^A - A)$ with $i \neq j \neq k \in C$. Since $\ell_i^A + \ell_{jk}^A - A = \min\{A, \ell_i\} + \min\{A, \ell_j + \ell_k\} - A = \min\{A, \ell_i, \ell_j + \ell_k, L - A\}$ with $L = \ell(C) \geq A$, we get $0 \leq \ell_i^A + \ell_{jk}^A - A \leq A$. It follows that

$$0 \leq \phi_0 \leq \frac{A}{2}.$$

It can be easily checked that both bounds are sharp. Notice that at the Shapley allocation, an insolvent firm ends up with a strictly positive payoff.

From system (11), the Shapley payment of creditor $i \in C$ is $\phi_i = \frac{1}{2}(A - \ell_{jk}^A) + \frac{1}{6}\sum_{j \neq i \in C}(\ell_{ij}^A - \ell_j^A) + \frac{1}{6}\ell_i^A$ with $i \neq j \neq k \in C$. Since $0 \leq A - \ell_{jk}^A = A - \min\{A, \ell_j + \ell_k\} = \max\{0, A - \ell_j - \ell_k\} \leq \ell_i^A$ and $0 \leq \ell_{ij}^A - \ell_j^A = \min\{A, \ell_i + \ell_j\} - \ell_j^A = \min\{A, \ell_i^A + \ell_j^A\} - \ell_j^A = \min\{A - \ell_j^A, \ell_i^A\} \leq \ell_i^A$, we get

$$\frac{\ell_i^A}{6} \leq \phi_i \leq \ell_i^A.$$

Again, all these bounds are sharp. For the debt forgiveness and for the truncated debt forgiveness of creditor $i \in C$ we immediately get the following sharp bounds:

$$\ell_i - \ell_i^A \leq \ell_i - \phi_i \leq \ell_i - \frac{\ell_i^A}{6} \qquad 0 \leq \ell_i^A - \phi_i \leq \frac{5\ell_i^A}{6}.$$

Notice that if for creditors $i \neq j$ we have $\ell_i \leq \ell_j$, then $0 \leq \ell_j^A - \ell_i^A \leq \ell_j - \ell_i$ and with the third creditor $k \neq i, j$, $0 \leq \ell_{jk}^A - \ell_{ik}^A \leq \ell_j^A - \ell_i^A \leq \ell_j - \ell_i$. Since $\phi_j - \phi_i =$

$\frac{1}{3}(\ell_j^A - \ell_i^A) + \frac{2}{3}(\ell_{jk}^A - \ell_{ik}^A)$, we get $0 \leq \phi_j - \phi_i \leq \ell_j^A - \ell_i^A \leq \ell_j - \ell_i$. It follows that at the Shapley allocation, the creditor with higher claim gets higher payment (unless $A \leq \ell_i, \ell_j$ when both get the same payment), but also gives an at least as high (truncated) debt forgiveness.

4 Properties of the Shapley allocation rule

In this section, we generalize the observations we made on the Shapley allocations for 2- and 3-creditor liability problems in Examples 11 and 12, and investigate further properties of the Shapley liability rule.

In Proposition 10, we showed that the Shapley rule satisfies efficiency, non-negativity and (truncated) liabilities boundedness, hence it is a liability rule. It immediately follows from these properties that the Shapley rule (as any liability rule) *respects minimal rights of creditors*, that is, it satisfies $\phi_i \geq \max\{0, A - \ell(C \setminus i)\}$ for any $i \in C$. Notice that the minimal right of creditor i is precisely his value $v(i)$ in the liability game, which is superadditive, and the Shapley value is well-known to prescribe individually acceptable payoffs in superadditive games. Recall that in Remark 9 we noticed that the Shapley rule (as any rule induced by a solution of an associated TU game) *ignores excessive parts of claims*, that is, $\phi(A, \ell) = \phi(A, \ell^A)$.

Since liability games are constant-sum, from sharing table (10), taken into account that $v(0 \cup S) = \ell_S^A$ for coalitions of the form $0 \cup S$ with $S \subseteq C$, we get that for liability problem (A, ℓ) the Shapley rule prescribes the following payments.

$$\phi_0(A, \ell) = -A + 2 \sum_{S \subseteq C} \gamma_n(s) \ell_S^A, \tag{12}$$

$$\phi_i(A, \ell) = 2 \sum_{S \subseteq C \setminus i} \gamma_n(s+1)(\ell_{S \cup i}^A - \ell_S^A), \qquad (i \in C) \tag{13}$$

where $s = |S|$ and $\ell_S^A = \min\{A, \sum_{i \in S} \ell_i\}$.

First, we establish lower and upper bounds for the Shapley payment of the firm.

Proposition 13. *Let $(A, \ell) \in \mathcal{L}^N$ be a liability problem and let v be the induced liability game on N. Then for the Shapley payment of the firm ϕ_0 we have that*

$$0 \leq \frac{n-2}{n} \min\{A, \min_{i \in C} \ell_i, \ell_C - A\} \leq \phi_0(A, \ell) \leq \frac{n-2}{n} A. \tag{14}$$

Proof. Since $v(0) = \ell^A_\emptyset = 0$, $v(N) = \ell^A_C = A$, and $\gamma_n(S) = \gamma_n(C \setminus S)$ for $S \subseteq C$,

$$\phi_0(A, \ell) \;=\; \sum_{\emptyset \neq S \neq C} \gamma_n(s)(\ell^A_S + \ell^A_{C\setminus S}) + \frac{2-n}{n}A. \tag{15}$$

If $n = 2$ then the summation in (15) is over the empty set, thus $\phi_0(A, \ell) = 0$. It means that the Shapley rule allocates the full asset value to the single creditor. In contrast, if $c \geq 2$, then the firm has some implicit bargaining leverage by threatening to form a coalition with the other creditors and compensate them first up to their full liabilities or the asset value. From $\ell^A_S + \ell^A_{C\setminus S} = \min\{2A, A + \ell_S, A + \ell_{C\setminus S}, \ell_C\} = A + \min\{A, \ell_S, \ell_{C\setminus S}, \ell_C - A\}$ and $\sum_{\emptyset \neq S \neq C} \gamma_n(s) = \frac{n-2}{n}$, where $s = |S|$, we get $\phi_0 = \sum_{\emptyset \neq S \neq C} \gamma_n(s) \min\{A, \ell_S, \ell_{C\setminus S}, \ell_C - A\}$. Equation (14) now follows. □

In the insolvent (non-degenerate) case, that is, if $\ell_C > A$ (and $A > 0$), $\min_{i\in C} \ell_i > 0$), the lower bound is positive, that is, the firm ends up with positive payoff. The lower bound in (14) is sharp if and only if $\ell_C - A \leq A$ and $\ell_C - A \leq \min_{i\in C} \ell_i$, that is, the deficiency of the firm does not exceed any of the individual liabilities and the asset value. The upper bound in (14) is sharp if and only if $A \leq \min_{i\in C} \ell_i$ (that implies $\ell_C - A \geq A$ for $c \geq 2$), that is, all creditors claim the full asset value so each one is willing to forgive some of its debt to stay a partner of the firm and receive some positive payment. Note that in this case as the number of creditors increases, the firm can keep almost all the asset value.

Second, we establish lower and upper bounds for the Shapley payments of the creditors.

Proposition 14. *Let $(A, \ell) \in \mathcal{L}^N$ be a liability problem and let v be the induced liability game on N. Then for any $i \in C$ for the Shapley payment of the creditor ϕ_i have that*

$$\frac{2}{n(n-1)}\ell^A_i \leq \phi_i(A, \ell) \leq \left(\frac{2}{n(n-1)} + \frac{(n-2)(n+1)}{n(n-1)}\right)\ell^A_i = \ell^A_i. \tag{16}$$

Proof. Since $v(0) = \ell^A_\emptyset = 0$ and $\gamma_n(1) = \frac{1}{n(n-1)}$, from (13) we get for $i \in C$,

$$\phi_i(A, \ell) = \frac{2}{n(n-1)}\ell^A_i + 2 \sum_{\emptyset \neq S \subseteq C\setminus i} \gamma_n(s+1)(\ell^A_{S\cup i} - \ell^A_S). \tag{17}$$

If $n = 2$, that is, $C = \{1\}$, then the summation in (17) is over the empty set, thus $\phi_1(A, \ell) = \ell^A_i$. It means that the Shapley rule allocates the full asset value to the single creditor. In contrast, if $c \geq 2$ then the summation in (17) is clearly non-negative, and it is zero if and only if $A \leq \ell_i^A$ for all $i \in C$. On the other side, $\ell_{S\cup i}^A - \ell_S^A = \min\{A - \ell_S^A, \ell_i^A\} \leq \ell_i^A$ in case of $A > \ell_S^A$. It follows from $\displaystyle\sum_{\emptyset \neq S \subseteq C\setminus i} \gamma_n(s+1) = \sum_{s=1}^{n-2} \binom{n-2}{s} \gamma_n(s+1) = \sum_{s=1}^{n-2} \frac{(n-2)!}{s!(n-2-s)!} \frac{1}{n} \frac{(s+1)!(n-2-s)!}{(n-1)!} = \sum_{s=1}^{n-2} \frac{s+1}{n(n-1)} = \frac{(n-2)(n+1)}{2n(n-1)}$ that the summation in (17) is at most $\dfrac{(n-2)(n+1)}{n(n-1)} \ell_i^A$, and equality holds if and only if $A \geq \ell_C$ (that implies $A \geq \ell_{S\cup i}$ for all $S \subseteq C \setminus i$). □

Both bounds are sharp in (16). The lower bound is attained when all creditors claim the full asset value, hence considerably weaken each other's bargaining position. On the other side, the creditors can be fully compensated if and only if the firm is solvent.

Next, we show that creditors with higher claims get higher Shapley payments, but they also give higher (truncated) debt forgiveness.

Proposition 15. *Let $(A, \ell) \in \mathcal{L}^N$ be a liability problem and v the induced liability game. Let $i, j \in C$ be such that $\ell_i \leq \ell_j$. At the Shapley value it holds that $\phi_i \leq \phi_j$, $\ell_i - \phi_i \leq \ell_j - \phi_j$ and $\ell_i^A - \phi_i \leq \ell_j^A - \phi_j$.*

Proof. Let $i, j \in C$ be two creditors with $\ell_i \leq \ell_j$, hence also $\ell_i^A \leq \ell_j^A$. Since liability games are constant-sum games, we use formula (2) to show $0 \leq \phi_j - \phi_i \leq \ell_j^A - \ell_i^A \leq \ell_j - \ell_i$.

When taking the difference $\phi_j - \phi_i$ the terms $v(S \cup i \cup j)$, $S \subseteq N \setminus \{i, j\}$, containing both players cancel out, so we get

$$\phi_j(v) - \phi_i(v) = \frac{2}{n} \sum_{s=0}^{n-1} \frac{1}{\binom{n-1}{s}} \sum_{S \subseteq N\setminus\{i,j\} : |S| = s} (v(S \cup j) - v(S \cup i)). \tag{18}$$

It is easily checked from the definition of v that $0 \leq v(S \cup j) - v(S \cup i) \leq \ell_j^A - \ell_i^A \leq \ell_j - \ell_i$ for all $S \subseteq N \setminus \{i, j\}$. Substituting each term in (18) with these non-negative constant bounds gives

$$0 \leq \phi_j(v) - \phi_i(v) \leq (\ell_j^A - \ell_i^A) \cdot \frac{2}{n} \sum_{s=0}^{n-1} \frac{1}{\binom{n-1}{s}} \binom{n-2}{s}, \tag{19}$$

since there are $\binom{n-2}{s}$ coalitions $S \subseteq N \setminus \{i, j\}$ of cardinality s. From $\frac{2}{n} \sum_{s=0}^{n-1} \frac{1}{\binom{n-1}{s}} \binom{n-2}{s} = \frac{2}{n} \sum_{s=0}^{n-1} (1 - \frac{s}{n-1}) = \frac{2}{n}(n - \frac{1}{n-1} \sum_{s=0}^{n-1} s) = \frac{2}{n}(n - \frac{n}{2}) = 1$, and the obvious $\ell_j^A - \ell_i^A \leq \ell_j - \ell_i$, the claim follows. □

The property formulated in Proposition 15 is called *order preservation* in the review article on bankruptcy rules by Thomson (2015). It obviously implies *equal treatment of* *equal creditors*, that is, if two creditors have the same claims, then they should get the same compensations. From Proposition 15 we readily get that the Shapley rule treats creditors with equal (truncated) liabilities in the same way.

Corollary 16. Let $(A, \ell) \in \mathcal{L}^N$ be a liability problem and v the induced liability game. Let $i, j \in C$ be such that $\ell_i = \ell_j$. At the Shapley value it holds that $\phi_i = \phi_j$, $\ell_i - \phi_i = \ell_j - \phi_j$ and $\ell_i^A - \phi_i = \ell_j^A - \phi_j$.

Next, we discuss monotonicity properties of liability rules. The question is how changes in the parameters of a liability problem influence the payments of the agents.

First, we investigate what happens to the payment of one creditor if his liability increases, but every other parameter of the problem stays put. We say that liability rule $f : \mathcal{L}^N \to \mathbb{R}_+^N$ is *liability monotonic* if for any creditor $i \in C$ and liability problems (A, ℓ), (A, ℓ') such that $\ell'_i > \ell_i$ and $\ell'_k = \ell_k$ for all $k \in C \setminus i$, it holds that $f_i(A, \ell') \geq f_i(A, \ell)$. We show that the Shapley rule is liability monotonic. Moreover, also the firm can only benefit from the increase of a liability.

Proposition 17. *Let liability problems (A, ℓ) and (A, ℓ') be such that $\ell'_i > \ell_i$ for $i \in C$, and $\ell'_k = \ell_k$ for all $k \in C \setminus i$. Then*

$$\phi_i(A, \ell') \geq \phi_i(A, \ell) + \frac{2}{n(n-1)} \min\{\ell'_i - \ell_i, A - \ell_i^A\}.$$

Moreover, $\phi_0(A, \ell') \geq \phi_0(A, \ell)$.

Proof. Let liability problems (A, ℓ) and (A, ℓ') be such that $\ell'_i > \ell_i$ for $i \in C$, and $\ell'_k = \ell_k$ for all $k \in C \setminus i$. Clearly, $\ell'^A_{S \cup i} \geq \ell^A_{S \cup i}$ and $\ell'^A_S = \ell^A_S$ whenever $S \subseteq C \setminus i$. From formula (17) we get

$$\phi_i(A, \ell') - \phi_i(A, \ell) = \frac{2}{n(n-1)}(\ell'^A_i - \ell^A_i) + 2 \sum_{\emptyset \neq S \subseteq C \setminus i} \gamma_n(s+1)(\ell'^A_{S \cup i} - \ell^A_{S \cup i}). \quad (20)$$

Since the summation term in (20) is non-negative, and $\ell'^A_i - \ell^A_i = \min\{\ell'_i - \ell_i, A - \ell^A_i\}$, the inequality for $\phi_i(A, \ell)$ follows.

From formula (12) we get

$$\phi_0(A, \ell') - \phi_0(A, \ell) = 2 \sum_{S \subseteq C \setminus i} \gamma_n(s+1)(\ell'^A_{S \cup i} - \ell^A_{S \cup i}) + 2 \sum_{S \subseteq C \setminus i} \gamma_n(s)(\ell'^A_S - \ell^A_S). \quad (21)$$

Since each term in the first summation is non-negative, and zero in the second one, we conclude that the payment to the firm can only increase if a liability increases. $\square$

Second, we investigate the changes in the payments to the creditors and the firm if the asset value increases, but all liabilities remain the same. We say that liability rule $f : \mathcal{L}^N \to \mathbb{R}^N_+$ is *asset monotonic for creditors* if for any creditor $i \in C$ and liability problems (A, ℓ), (A', ℓ) such that $\ell(C) \geq A' > A$, it holds that $f_i(A', \ell) \geq f_i(A, \ell)$. We show that the Shapley rule is asset monotonic for creditors, but the firm can end up with smaller or with higher payoff.

Proposition 18. *Let liability problems (A, ℓ) and (A', ℓ) be such that $\ell(C) \geq A' > A$. Then for any creditor $i \in C$,*

$$0 \leq \phi_i(A', \ell) - \phi_i(A, \ell) \leq \min\{A' - A, \ell_i\},$$

and for the firm,

$$\frac{2-n}{n}(A' - A) \leq \phi_0(A', \ell) - \phi_0(A, \ell) \leq \frac{n-2}{n}(A' - A).$$

Moreover, for $c = |C| \geq 2$, all bounds are sharp.

In case of a single creditor $C = \{1\}$, $\phi_1(A', \ell) - \phi_1(A, \ell) = A' - A$ and $\phi_0(A', \ell) = \phi_0(A, \ell)$.

Proof. Let liability problems (A, ℓ) and (A', ℓ) be such that $\ell(C) \geq A' > A$. From formula (13) we get for any $i \in C$,

$$\phi_i(A', \ell) - \phi_i(A, \ell) = 2 \sum_{S \subseteq C \setminus i} \gamma_n(s+1) \left[(\ell^{A'}_{S \cup i} - \ell^{A}_{S \cup i}) - (\ell^{A'}_{S} - \ell^{A}_{S}) \right]. \tag{22}$$

First of all, since $\left[(\ell^{A'}_{S \cup i} - \ell^{A}_{S \cup i}) - (\ell^{A'}_{S} - \ell^{A}_{S})\right] = \left[(\ell^{A'}_{S \cup i} - \ell^{A'}_{S}) - (\ell^{A}_{S \cup i} - \ell^{A}_{S})\right]$ and the difference $\ell^{A}_{S \cup i} - \ell^{A}_{S} = \min\{\ell_i, A - \ell^{A}_{S}\}$ where $A - \ell^{A}_{S} = \max\{A - \ell_S, 0\}$ is clearly non-decreasing in A, we get that the difference in the bracket in each term is non-negative, implying asset monotonicity for creditor $i \in C$.

Let us assume $c \geq 2$. Then there are at least two different terms in (22). One is the term for $S = \emptyset$. It equals $\frac{2}{n(n-1)} \left[(\ell^{A'}_{i} - \ell^{A}_{i}) - (0 - 0)\right]$. The difference in the bracket can range from 0 (attained, if $\ell_i \leq A < A'$) to $\min\{A' - A, \ell_i\}$ (attained, if $A < A' \leq \ell_i$). The other term is for $S = C \setminus i \neq \emptyset$. It equals $\frac{2}{n} \left[(\ell^{A'}_{C} - \ell^{A}_{C}) - (\ell^{A'}_{C \setminus i} - \ell^{A}_{C \setminus i})\right] = \frac{2}{n} \left[(A' - A) - (\ell^{A'}_{C \setminus i} - \ell^{A}_{C \setminus i})\right]$. Again, the difference in the bracket can range from 0 (attained, if $A < A' \leq \ell_{C \setminus i}$) to $(A' - A)$ (attained, if $\ell_{C \setminus i} \leq A < A'$)). Likewise, if $\ell_i \leq A < A'$ but $A < A' \leq \ell_j$ for any other creditor $j \neq i$, then all terms in (22) are zero, implying that the zero lower bound is indeed sharp. In contrast, if $A < A' \leq \ell_i$ but $\ell_{C \setminus i} \leq A < A'$ (implying $\ell_j \leq A < A'$ for any other creditor $j \neq i$), then the differences in all brackets in (22) are equal to $\min\{A' - A, \ell_i\}$. In light of $2\sum_{S\subseteq C\setminus i}\gamma_n(s+1) = 1$, the claimed upper bound is also sharp.

For the change in the Shapley payment to the firm, taken into account that $\ell_\emptyset^A = 0$ and $\ell_C^A = A$, from formula (12) we get

$$\phi_0(A', \ell) - \phi_0(A, \ell) = -(A' - A) + 2\sum_{\emptyset\neq S\subsetneq C}\gamma_n(s)(\ell_S^{A'} - \ell_S^A) + \frac{2}{n}(A' - A). \tag{23}$$

Since the difference $\ell_S^{A'} - \ell_S^A$ is clearly non-negative but cannot exceed $A' - A$, from $\sum_{\emptyset\neq S\subsetneq C}\gamma_n(s) = 1 - \frac{2}{n}$, the claimed inequalities for the difference $\phi_0(A', \ell) - \phi_0(A, \ell)$ follow. The negative lower bound is attained if $\ell_S \leq A$ for every non-empty set of creditors $S \neq C$ implying $\ell_S^{A'} - \ell_S^A = 0$. The positive upper bound is attained if $\ell_i \geq A'$ for all creditors $i \in C$ implying $\ell_S \geq A'$ and $\ell_S^{A'} - \ell_S^A = A' - A$ for every non-empty set of creditors $S \neq C$.

Finally, in case of a single creditor $C = \{1\}$, equation (22) simplifies to $\phi_1(A', \ell) - \phi_1(A, \ell) = \frac{2}{2(2-1)}\left[(\ell_i^{A'} - \ell_i^A) - (0 - 0)\right] = A' - A$, reconfirming that the Shapley rule gives everything to the single creditor. By efficiency, the firm ends up with nothing, thus, $\phi_0(A', \ell) - \phi_0(A, \ell) = 0 - 0 = 0$. Notice that for $n = 2$, the summation in (23) is over the empty set, and the claimed lower and upper bounds coincide at zero. $\square$

The following property, called super-modularity by Thomson (2015), is a kind of combination of order preservation (when the payments to two creditors in the same problem are compared) and asset monotonicity (when the payments to the same creditor in two related problems are compared). We say that liability rule $f : \mathcal{L}^N \to \mathbb{R}_+^N$ is *super-modular for creditors* if for any two creditors $i, j \in C$ with $\ell_i \geq \ell_j$ and liability problems (A, ℓ), (A', ℓ) such that $\ell(C) \geq A' > A$, it holds that $f_i(A', \ell) - f_i(A, \ell) \geq f_j(A', \ell) - f_j(A, \ell)$. We show that the Shapley rule is super-modular for creditors, thus it allocates from the increment in the asset value more to creditors with higher liabilities.

Proposition 19. *Let liability problems (A, ℓ) and (A', ℓ) be such that $\ell(C) \geq A' > A$. If $\ell_i \geq \ell_j$ for creditors $i, j \in C$ then*

$$0 \leq (\phi_i(A', \ell) - \phi_i(A, \ell)) - (\phi_j(A', \ell) - \phi_j(A, \ell)) \leq \min\{\ell_i - \ell_j; A' - A\}. \tag{24}$$

Proof. Given two creditors $i, j \in C$, a set of creditors $S \subseteq C$ can be one of four types: S contains both i and j; contains i but not j; contains j but not i; contains neither i nor j. For brevity, we represent $S \subseteq C$ respectively as Rij, Ri, Rj, R with a generic

$R \subseteq C \setminus \{i,j\}$. From the formula in (13) we get that $\phi_i(A',\ell) - \phi_i(A,\ell) =$

$$2\sum_R \left\{\gamma_n(r+1)\left[\ell^{A'}_{Ri} - \ell^{A}_{Ri} - \ell^{A'}_{R} + \ell^{A}_{R}\right] + \gamma_n(r+2)\left[\ell^{A'}_{Rji} - \ell^{A}_{Rji} - \ell^{A'}_{Rj} + \ell^{A}_{Rj}\right]\right\}. \quad (25)$$

Exchanging i and j gives $\phi_j(A',\ell) - \phi_j(A,\ell) =$

$$2\sum_R \left\{\gamma_n(r+1)\left[\ell^{A'}_{Rj} - \ell^{A}_{Rj} - \ell^{A'}_{R} + \ell^{A}_{R}\right] + \gamma_n(r+2)\left[\ell^{A'}_{Rij} - \ell^{A}_{Rij} - \ell^{A'}_{Ri} + \ell^{A}_{Ri}\right]\right\}. \quad (26)$$

Subtracting (26) from (25) gives $(\phi_i(A',\ell) - \phi_i(A,\ell)) - (\phi_j(A',\ell) - \phi_j(A,\ell)) =$

$$2\sum_R [\gamma_n(r+1) + \gamma_n(r+2)]\left[(\ell^{A'}_{Ri} - \ell^{A}_{Ri}) - (\ell^{A'}_{Rj} - \ell^{A}_{Rj})\right]. \quad (27)$$

Suppose $\ell_i \geq \ell_j$, implying $\ell_{Ri} \geq \ell_{Rj}$. It is easily checked that

$$(\ell^{A'}_{Ri} - \ell^{A}_{Ri}) - (\ell^{A'}_{Rj} - \ell^{A}_{Rj}) = \begin{cases} 0, & \text{if } \ell_{Rj} \leq \ell_{Ri} \leq A \leq A', \\ \ell_{Ri} - A, & \text{if } \ell_{Rj} \leq A \leq \ell_{Ri} \leq A', \\ A' - A, & \text{if } \ell_{Rj} \leq A \leq A' \leq \ell_{Ri}, \\ \ell_{Ri} - \ell_{Rj}, & \text{if } A \leq \ell_{Rj} \leq \ell_{Ri} \leq A', \\ A' - \ell_{Rj}, & \text{if } A \leq \ell_{Rj} \leq A' \leq \ell_{Ri}, \\ 0, & \text{if } A \leq A' \leq \ell_{Rj} \leq \ell_{Ri}. \end{cases}$$

It follows that

$$0 \leq (\ell^{A'}_{Ri} - \ell^{A}_{Ri}) - (\ell^{A'}_{Rj} - \ell^{A}_{Rj}) \leq \min\{\ell_{Ri} - \ell_{Rj} = \ell_i - \ell_j; A' - A\}.$$

Taken into account that

$$\begin{aligned} \sum_{R \subseteq C \setminus ij} [\gamma_n(r+1) + \gamma_n(r+2)] &= \sum_{R \subseteq C \setminus ij} \gamma_n(r+1) + \sum_{j \in Q \subseteq C \setminus i} \gamma_n(q+1) \\ &= \sum_{S \subseteq C \setminus i} \gamma_n(s+1) = 1/2, \end{aligned}$$

where $q = |Q|$ and $s = |S|$, from (27) we get the claimed inequalities in (24). $\square$

A straightforward corollary of Proposition 19 is that if $\ell_i = \ell_j$ for creditors $i, j \in C$ then

$$\phi_i(A',\ell) - \phi_i(A,\ell) = \phi_j(A',\ell) - \phi_j(A,\ell).$$

5 Complexity of computing the Shapley value

Even though liability games are constant-sum and we showed in (12) and (13) that the Shapley value can be calculated directly from a liability problem, now we prove that calculating the Shapley payoff to the firm is NP-hard.

Theorem 20. *Given two liability problems and the induced liability games, it is NP-hard to verify whether the firm has the same Shapley value in both games.*

Proof. Recall the NP-complete subset sum problem SUBSUM (See for instance Garey and Johnson (1979)): given $a_1, a_2, \ldots, a_n \in \mathbb{Z}$ and $K \in \mathbb{Z}$ we ask whether there exists a subset $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ such that $\sum a_{i_j} = K$. Here we consider a special case of this problem: HALFSUM: given positive integers $a_1, a_2, \ldots, a_n$ we ask whether there exists a subset $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ such that $\sum a_{i_j} = \frac{\sum a_i}{2}$. It is very easy to show by the following steps that HALFSUM is still NP-complete.

- It is trivial to show that SUBSUM is NP-complete if we restrict it to even numbers, so we can assume that $\sum a_i$ is even.
- We get an equivalent instance of SUBSUM if we replace K by $\sum a_i - K$. Using this observation, it is clear that we can assume that $K \leq \frac{\sum a_i}{2}$.
- This special form of SUBSUM can be reduced to HALFSUM by adding an extra number $a_{n+1} = \frac{\sum a_i}{2} - K$ to the set.

We reduce HALFSUM to the Shapley value calculation. Let $HS = (a_1, a_2, \ldots, a_n)$ be an instance of the HALFSUM problem. Consider the liability problems (A, ℓ) and $(A-1, \ell)$, where $\ell = (\ell_1, \ell_2, \ldots, \ell_n) = (a_1, a_2, \ldots, a_n)$ and $A = \frac{\sum a_i}{2}$. Let v and v_2 be the liability games corresponding to (A, ℓ) and $(A-1, \ell)$, respectively. We show that the defaulting firm has a different Shapley value in v and v_2 if and only if the instance of the HALFSUM problem has a solution.

Given a subset of creditors $S \subseteq C$, let $\mathrm{mc}(S) = v(S \cup \{0\}) - v(S)$ be the marginal contribution of player 0 in the liability game v, corresponding to the first liability problem. We claim that

$$\mathrm{mc}(S) = \begin{cases} \ell(S), & \text{if } \ell(S) \leq A, \\ \ell(C \setminus S), & \text{if } \ell(S) \geq A. \end{cases} \tag{28}$$

To prove (28), recall that the value of the assets A is exactly half of the sum of liabilities. Notice that creditors in S can be paid if and only if creditors in $C \setminus S$ cannot be paid. If $\ell(S) \leq A$, then $v(S) = 0$, however, in this case $v(S \cup \{0\}) = \ell(S)$. If $\ell(S) \geq A$, then $v(S) = A - \ell(C \setminus S)$ and $v(S \cup \{0\}) = A$.

Let ϕ_0 be the Shapley value of player 0 in v. We have that

$$n!\phi_0 = \sum_{S \subseteq C} |S|!(n - |S| - 1)!\mathrm{mc}(S) = \sum_{\ell(S) < A} |S|!(n - |S| - 1)!\ell(S)$$
$$+ \sum_{\ell(S) = A} |S|!(n - |S| - 1)!A + \sum_{\ell(S) > A} |S|!(n - |S| - 1)!\ell(C \setminus S). \quad (29)$$

Now consider the game v_2, that is, decrease the asset value A by 1. Let $\mathrm{mc}_2(S) = v_2(S \cup \{0\}) - v_2(S)$.

If S is a coalition such that $\ell(S) < A$, then $\ell(S) \leq A - 1$, so the liabilities in S can still be paid in v_2 and $\ell(C \setminus S) > A > A - 1$, liabilities in $C \setminus S$ obviously cannot be paid with less asset value. It follows that $v_2(S) = 0$ and $v_2(S \cup \{0\}) = \ell(S)$. (Recall that ℓ is the same in both problems.) Now let's consider a coalition of creditors $S \subset C$ such that $\ell(S) > A$. In this case $\ell(C \setminus S) < A$, that is, $\ell(C \setminus S) \leq A - 1$. Liabilities in S cannot be paid and liabilities in $C \setminus S$ can be paid not only in game v but also in game v_2. This means that $v_2(S) = A - 1 - \ell(C \setminus S)$ and $v_2(S \cup \{0\}) = A - 1$, so $\mathrm{mc}(S) = (A - 1) - (A - 1 - \ell(C \setminus S) = \ell(C \setminus S)$.

It follows that in (29), the first and the last term do not change in v_2, implying that if *HS* is a FALSE instance of problem HALFSUM, then the sum of these terms does not change when we decrease the value of assets by 1. In this case, the second term is empty.

On the other hand, let's consider a coalition where $\ell(S) = A$ exactly. In this case, $v(S) = 0$ and $v(S \cup \{0\}) = \mathrm{mc}(S) = A$ in the first game. However, in the second game, $v_2(S) = v(S) = 0$ but $v_2(S \cup \{0\}) = \mathrm{mc}_2(S) = A - 1$. If *HS* is a TRUE instance of the HALFSUM problem, then the Shapley value of player 0 decreased in game v_2 compared to game v. □

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