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John von Neumann's game-theoretic legacy

ANDRÁS SIMONOVITS

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ABSTRACT
John von Neumann (Budapest, 1903–Washington D.C., 1957) was an exceptional polymath, who made fundamental contributions to mathematical logics, functional analysis, quantum mechanics, game theory, computer architecture and automata theory. In this brief paper, I shall review the game-theoretic results of von Neumann and their legacy in an informal way.

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András Simonovits
CERS IE and BME MI
simonovits.andras@krtk.-hun-ren.hu

Neumann János játékelméleti öröksége

SIMONOVITS ANDRÁS

ÖSSZEFOGLALÓ

Neumann János (1903, Budapest—1957, Washington, D,C) kivételes polihisztor volt, aki alapvetően továbbfejlesztette a matematikai logikát, a funkcionálanalízist, a kvantummechanikát, a játékelméletet, a számítógéparchitektúrát és az automataelméletet. Ebben a rövid tanulmányban Neumann János játékelméleti eredményeit és örökségét informális módon tekintem át.

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Kulcsszavak: játékelmélet

John von Neumann's game-theoretic legacy

András Simonovits
email: simonovits.andras@krtk.hun-ren.hu
CERS IE, BME MI
Budapest, Tóth Kálmán u 4, 1097, Hungary

John von Neumann (Budapest, 1903–Washington D.C., 1957) was an exceptional polymath, who made fundamental contributions to mathematical logics, functional analysis, quantum mechanics, game theory, computer architecture and automata theory. In this brief paper, I shall review the game-theoretic results of von Neumann and their legacy in an informal way.

1. Games

Parlor games

Commonly speaking, many people like to play games like chess, bridge, etc. At first sight, it is not evident why these games deserve mathematical treatment. But their study by von Neumann and others revolutionized the field of social sciences in general and economics in particular. In the framework of these sciences, game theory analyzes situations where there are at least two decision makers, each choosing among her potential decisions influencing not only her own utility but also others'.

Chess is probably the best-known example for parlor games. This game was already mathematically studied by Zermelo (1912). There are two players (White and Black) who choose their steps according to a given set of rules, and depending on the sequences of their feasible steps, there are three outcomes: White wins, Black wins, the game ends in a draw. One important step in the analysis of such games was to compress the sequences of these steps into strategies as done in chess books and then consider the payoffs or utilities of such strategies.

In the case of chess, the White can get 1, 1/2 or 0 point, and correspondingly, the Black gets 0, 1/2 and 1 point. This is a constant-sum game (the sum is equal to 1), but game theorists transform it into a zero-sum game by deducting 1/2 from the usual points ending with 1/2, 0 and –1/2 or after multiplication by 2, 1, 0 and –1. Chess is a very complicated game and even more than hundred years after Zermelo had published his paper, we do not know much more than he knew: one of the players can always win or at least reach a draw; but we still do not know whether White or Black.

Two-player zero-sum games

Following von Neumann, we can now define abstract two-player zero-sum games, where there are two players (numbered 1 and 2). Player 1 can choose a strategy s_1 from a set S_1 and player 2 can choose a strategy s_2 from a set S_2. Player 1's utility is a scalar, depending on both strategies: $u(s_1, s_2)$; and player 2's utility is its negative: $-u(s_1, s_2)$. In such a situation, we cannot simply define maximization, we have to be satisfied with analyzing an equilibrium. But how to define an *equilibrium* in such a game? For example, can the Black achieve at least a draw?

Anticipating a later and simpler formulation by Nash (1951), we shall avoid the quite sophisticated minimax definition. We say that a pair of strategies (s_1^*, s_2^*) form an equilibrium if neither player can increase her utility by deviating from her equilibrium if the other player sticks to his equilibrium. Mathematically,

$$u(s_1^*, s_2^*) \geq u(s_1, s_2^*) \text{ and } -u(s_1^*, s_2^*) \geq -u(s_1^*, s_2).$$

Multiplying both sides of the second inequality by –1 yields

$$u(s_1, s_2^*) \leq u(s_1^*, s_2^*) \leq u(s_1^*, s_2).$$

This is the well-known saddle-point inequality, and it can already be formulated as the equality of minmax and maximin values as was done by Neumann (1926).

In a mathematical theory, it is not sufficient to define an equilibrium (it is possible that more than one equilibrium exists), one must show that at least one equilibrium exists. To understand the mathematical problem, we shall start with a trivial example: *matching pennies.* Two players choose independently Head or Tail. Player 1 (she) gains Player 2's (his) penny if they choose the same side, and she loses her own penny to her opponent if they choose different sides. It is easy to see that in this game, if the players played predetermined (pure) strategies, then there would not exist any *equilibrium*. To hide their intentions, both players *randomize* their pure strategies, i.e., they choose Head or Tail with certain probabilities independently. We assume that each player wants to maximize her/his *expected* gain/loss, i.e., weighting the gains and losses by the probabilities of the four events HH, HT, TH and TT.

Here the equilibrium is a pair of probabilities of choosing H (and T), where no player can gain by *unilaterally* deviating from her/his own strategy. Evidently, in this generalized game, there exists a unique equilibrium; namely, each player tosses her/his own penny, and accepts the random result as her/his choice. Statistically, both players choose H and T with probability 1/2–1/2 and each player's expected utility is equal to zero.

Of course, such a model is only relevant if the game is played many occasions or by many players. Then the randomness is achieved by the distribution of the pure strategies, i.e. 50-50% of both types of players chose either H or T etc.

Von Neumann considered general two-person zero-sum games where the numbers of pure strategies might be any finite number rather than 2, and the payoff structure is general. The most important result of von Neumann (1928) reads as follows: *In any two-player zero-sum finite game, there exists at least a mixed equilibrium*.

John von Neumann

Oskar Morgenstern

Expected utility

For a long time, von Neumann (1928) has hardly been noticed. (Another outstanding mathematician, Emile Borel's related and earlier works, see Borel, 1921, also remained unnoticed.) By the time von Neumann and Morgenstern (1944, 1947) published their monograph, *Theory of Games and Economic Behavior*, von Neumann had already become a superstar in the US military–scientific establishment and their book was a success.

Here we shall only allude to another outstanding contribution by von Neumann (and Morgenstern), namely their axiomatic treatment of *expected utility* in the second edition of their monograph. Note that in calculating the expected utility of matching pennies, we implicitly assumed that the gains/losses enter linearly. It is not obvious, however, if the gains and the losses always enter this way in the utility. By axiomatizing the consistent decision making with generalized lotteries, von Neumann and Morgenstern revolutionized the theory of utility and generated a still lasting controversy, to be discussed later on.

Other achievements

It is to be mentioned that von Neumann and Morgenstern (1944) also introduced *cooperative games*, where the players can make contracts between various coalitions (subsets of players).

Though only weakly related to game theory, von Neumann (1937)'s *multisectoral growth model* should be mentioned. Here von Neumann used a fixed-point theorem first in social sciences, opening a new route. (A function f mapping an n-dimensional set into itself, has a fixed point x^* if $f(x^*) = x^*$.) This model was also a forerunner to the input-output models (initiated by Leontief, 1941) and linear programming from 1947 (for a summary, Dantzig, 2002). After World War II, various schools of game theory developed in the USA around von Neumann.

2. Post-Neumann developments in game theory

We shall now discuss several developments following the publication of von Neumann and Morgenstern (1944, 1947): prisoners' dilemma, Nash-equilibrium, Shapley-value, Harsányi's generalization of games with perfect information and others.

Prisoners' dilemma

Perhaps the *prisoners' dilemma* is the best-known non-zero-sum game. The story can be formulated as follows. Two connected criminals are imprisoned on weak evidence and the police need the help of at least one prisoner to testify against the other. The prisoners are separated and can choose independently of the other between testify or not. Table 1 displays the joint payoffs.

Table 1. Joint payoff in the prisoners' dilemma

Player 2 Player 2	Betray	Not
Betray	(–2, –2)	(3, –3)
Not	(–3, 3)	(2, 2)

Regardless of how the other criminal decides, each criminal is better-off if he betrays his partner than if he does not (betray is a *dominant* strategy). Indeed, if Player 2 betrays Player 1, Player 1 gains if she also betrays: –2 > –3 (column 1). If Player 2 does not betray Player 1, Player 1 gains if she betrays him: 3 > 2 (column 2). In this example, the equilibrium is simply (betray, betray). Note that if they could cooperate with each other, then both would be better-off. In more general situations, the definition of the equilibrium is much more difficult.

Nash-equilibrium

At the same time, John Nash (1951) recognized that the assumptions of two players and of zero-sum utilities are much more restrictive than von Neumann presumed. He introduced a general model with *n* (>1) players and quite general, unconnected utility functions. (In Neumann, $u_1=-u_2$!) Thus, he removed the barrier in the applications of game theory to more general situations. But he needed a definition of the equilibrium valid beyond the narrow scope of the zero-sum games or the dominant strategy of the prisoners' dilemma. By his definition, an equilibrium is a vector formed by the players' appropriate strategies if no player can raise its payoff deviating from the equilibrium *unilaterally*. For simplicity, we present the definition of a Nash-equilibrium for a 2-player game with general utility functions, changing Neumann's u to u_1 and $-u$ to u_2 in the above definition of equilibrium:

$$u_1(s_1^*, s_2^*) \geq u_1(s_1, s_2^*) \text{ and } u_2(s_1^*, s_2^*) \geq u_2(s_1^*, s_2).$$

Under quite general assumptions, the existence of a Nash-equilibrium can be proven for any finite number of players.

To understand the relevance of the *one-sided* deviation, note that in the prisoners' dilemma, if both players deviate from the equilibrium, then both can improve their lots.

Obviously, Nash's result was a pathbreaking generalization of von Neumann's original problem. It is to be noted, however, that when the Ph.D. student Nash presented his discovery to von Neumann, the superstar did not recognize its merit and only coolly remarked: the proof depends on the fixed-point theorem already used in von Neumann (1938). [Indeed, defining the *best response* of player 1 to her partner's strategy s_2 as $b_1(s_2)$ and the best response of player 2 to his partner's s_1 as $b_2(s_1)$, any Nash-equilibrium is a fixed point of the composite function (b_1, b_2), since s_1*= $b_2(s_1$*) and s_2*= $b_2(s_2$*).]

John Nash Lloyd Shapley

As a real-life example of Nash-setup, look at an *oligopolistic* market where few great firms compete on the same market, forming a middle point between a monopoly and perfect competition. (For example, in the 1950s, three great US carmaker firms, GM, Ford and Chrysler competed producing similar cars.) Consider now n>1 similar competing firms with potentially different amounts of output. It is assumed that the greater the total supply, the lower the equilibrium price. Each firm chooses its output to maximize its profits, without knowing what the other firms do. Each firm can increase its profit by raising its output if the other firms accommodate. Nash's result implies the existence of a Nash-equilibrium with the following feature: The more firms *compete*, the *higher* the total output and the *lower* the equilibrium price, asymptotically converging to the unit costs. (It is of interest that an otherwise famous French economist, Cournot had solved the duopolium problem as early as 1838 but game theorists had not paid attention to his solution until the 1950s.)

Cost-sharing as a cooperative game

Moving from noncooperative to *cooperative games*, we consider n players, who can make contracts within *coalitions* (subsets of the set of all players). The game is defined by the coalitional payoffs, freely transferable. It is assumed that any coalition can get at least as much together as the sum of the members' individual payoffs. For example, two cooperating workers can move a heavy object from A to B which they cannot move individually. Von Neumann and Morgenstern formulated their concept of cooperative solution but it was too complicated; moreover, its existence was uncertain. In contrast, another concept, later called *Shapley-value* (Shapley, 1953) proved to be a very useful concept for finding a cooperative equilibrium.

Consider another simple example, *cost sharing* on overlapping taxi routes. Assume that A wants to ride a taxi from point 0 to point 2, while B wants to travel only from 1 to 2; points 0, 1, and 2 being the x-coordinates of three points on the horizontal line. How to share the taxi costs if A and B are ready to travel together? If cost-sharing is to be proportional to *distances* (like in travelling by train), then the total cost is shared as 2/3, 1/3. Shapley chose another, *axiomatic* approach, which in our taxi example implies that A pays the full cost of the trip from 0 to 1 and shares equally the cost of ride from 1 to 2. Then Shapley value shares the cost as follows:

$$a=(1+1/2)/2=3/4 \text{ and } b=(1/2)/2=1/4.$$

This simple example can be generalized to an arbitrary joint cost function and other problems. Note the difference between riding a train or riding a taxi!

The *core* is another equilibrium concept, a set of feasible allocations, where no coalition can improve its total share by leaving the grand coalition. For example, in our earlier example of moving a heavy object, denote the value of moving the object by 1. The core consists of the following allocations: worker 1 gets x, and worker 2 gets $1—x$ where $0 < x < 1$.

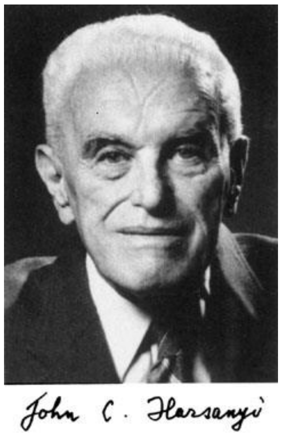
Unknown utility functions and unknown strategy spaces

What happens to Nash-equilibrium if players *do not know* each other's utility functions or possibilities? Vickrey (1960) considered the following simple example. There are n potential buyers who independently *bid* on the same picture and their individual values are uniformly and independently distributed on a common interval. A Bayesian Nash-equilibrium (BNE) is a generalization of the traditional Nash-equilibrium, where each bidder maximizes her expected gain, assuming that the others do the same and having a Bayesian probability distribution concerning the others' characteristics. It is irrational to bid above the individual value, and each bidder has to choose his bid as to raise the probability of winning and the difference between the individual value and the bid. Then in the BNE, the individual bid is equal to $(n–1)/n$ times the individual value.

Numerical *illustration*: when 10 potential buyers make bids, then in the BNE, everybody bids 90% of her individual value. The example has a simple moral: the more bidders compete, the more `honest' they are.

John Harsanyi (1967) generalized this and other models into a *general* Bayesian framework for games of incomplete information. Here players assume that there is a probability distribution on the utility functions and strategy sets, and BNE applies.

William Vickrey



John C. Harsanyi

Alternatives to expected utility

Though most game theorists neglect the question of expected utility functions, a number of powerful economists, including Nobel-prize winners, reformulated it. Already Friedman and Savage (1948) proposed alternatives. Then the somewhat eccentric Maurice Allais showed that a basic axiom is not satisfied in practice (Allais, 1953). In their prospect theory, Kahneman and Tversky (1979) proposed a psychologically more realistic approach (see also Kőszegi and Rabin, 2006).

If the value of the 'penny' is huge, then the eventual loss is not compensated for an equal potential gain. On the other hand, there are games where the gains are more important than the losses are. Here we only outline a very simple solution. Consider a pure chance game, where a player buys one ticket for C = \$1 and a machine returns her either G = \$0.23/p with a known probability p or zero dollar with probability $1–p$. The expected monetary gain is \$0.23, hence on balance the player loses \$0.77. If p is large (close to 1), then nobody plays the game; but if p is small (close to 0), then it may be attractive to play it: the almost sure loss is negligible but the improbable gain is huge. Of course, this is true for the various versions of lottery as well.

Skipping the intricate axiomatization of von Neumann and Morgenstern (otherwise criticized by the Nobel-prize laurate Allais, 1953), we assume that at the start of the game, the player has an endowment W = \$10. We shall work with a Bernoulli utility function of $u(x)=x^4$, where x denotes the player's relative wealth after the trial: $1+(G–C)/W$ or $1–C/W$ or staying away: 1. The expected utility is the weighted sum of the utilities of winning and losing. Columns 1, 2 and 3 of Table 2 respectively display five small winning probabilities, the resulting prizes and the expected utility of playing. Note that if the numerical utility is higher/lower than 1, then a rational player participates/stays away. The critical probability of being indifferent is a little bit below 0.02.

Table 2. Winning probability and expected utility

Probability of winning	Prize, $	Expected utility
0.01	23.000	1.698
0.02	11.500	0.996
0.03	7.667	0.868
0.04	5.750	0.819
0.05	4.600	0.794

Contributions of other Nobel prize winners to game theory

Here we only mention few economists who also received Nobel Memorial Prizes in Economic Sciences for their contributions to game theory.

The multiplicity of Nash equilibria may be problem, the more so, that unlike in zero-sum games, the elements of different Nash equilibria are not interchangeable. (Example: keep right or keep left are Nash equilibria but their mix is not.) From 1965 on, Reinhard Selten introduced so-called *refinements* to reduce the excessive number of Nash-equilibria (see Selten, 1988).

To understand the behavior of nonoptimal players, biologist John Maynard Smith discovered *evolutionary stable strategies* (see Maynard Smith and Price, 1973), defined as follows: if coexisting types play such strategies, then a local deviation from it is punished.

Among a lot of important social and political issues, Thomas Schelling (Schelling, 1971) studied dynamic *racial segregation* in US cities. Rather than being satisfied by an unrealistic analytical model, he used a primitive but relevant agent-based model where pennies and dimes moved on the board if they were unhappy with the color of their neighbors. He showed that an apparently a racially mixed distribution can easily degenerate into a racially separated one.

The proof of existence and the characterization of *competitive equilibria* have been a central task of economics. In the simplest setup, a set of allocation of goods among consumers and a set of prices form a competitive equilibrium if the allocation is balanced, under these prices the exchanges are financially feasible; and consumers maximize their utilities. Introducing a continuum of players, Robert Aumann proved that the core (mentioned above) and the *set* of competitive equilibria are the same (Aumann, 1964).

Starting with wives and husbands and continuing with students and universities, *pairing couples* is a recurrent topic in social life. From the 1980s, Alvin E. Roth and Lloyd S. Shapley applied the theory of stable allocations to market design: for example, pairing couples in simultaneous *kidney exchange* (see e.g., Roth and Sotomayor, 1990).

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